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A kinetic description of anisotropic fluids with multivalued internal energy

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Abstract

This work is concerned with some extensions of the classical compressible Euler model of fluid dynamics in which the fluid internal energy is a measure-valued quantity. A first extension was derived from the hydrodynamic limit of a kinetic model involving a specific class of collision operators typical from quasi-linear plasma theory (see Eur. J. Mech. B Fluids 20 (2001) 303–327, and Contin. Mech. Thermodyn. 10 (1998) 153–178). In these papers the collision operator simply describes the isotropization of the kinetic distribution function about some averaging velocity. In the present work we introduce a new extension of such models in which the relaxed distribution is anisotropic. Similarly to (Eur. J. Mech. B Fluids 20 (2001) 303–327) and (Contin. Mech. Thermodyn. 10 (1998) 153–178) this model is derived from a kinetic equation with a collision operator that relaxes to anisotropic equilibria. We then investigate diffusive corrections of this fluid-dynamical model using Chapman–Enskog techniques and show how the anisotropic character affects the expression of the viscosity and of the heat flux.

Keywords: Boltzmann equation; Euler equation; Navier–Stokes equation; Viscosity; Heat conduction; Turbulence; Anisotropic fluid; Chapman–Enskog expansion

1. Introduction

Anisotropic fluid models are widely used in relativistic astrophysics and cosmology. This has motivated increasing interest and progress in the study of (relativistic) fluid systems with anisotropic stresses in the last years. From a physical point of view, anisotropic fluid spheres have constituted a useful model for discussing anisotropy since the early 70's (see [1]). More recently, the spherically symmetric collapse of anisotropic fluid objects into a black hole has been investigated by several authors [2,3]. Actually, there are known exact, analytical solutions of anisotropic fluid bodies collapsing into black holes. Also, theoretical investigations about realistic stellar models show that the stellar matter is composed of anisotropic fluid at least in very high density ranges, where nuclear interactions are dealt with relativistically. An important example is solar wind. Indeed, the electromagnetic turbulence in solar wind tends to be anisotropic, with smooth variations along the ambient magnetic field and sharp variations perpendicular to the ambient field. Other fields of application of anisotropic fluid theories are liquid crystals, fiber composites, flow through porous media and electrorheology. In fact, a great part of the potential applicability of anisotropic fluid theory comes from complex materials with internal microstructure such as polymers, melts, concentrated suspensions and emulsions, among others.

In this paper we examine a kinetic approach to anisotropic fluids with multivalued internal energy from a mathematical perspective. In a recent work [4], a fluid-dynamical model which extends the classical Euler equations of compressible gas

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dynamics was investigated. It consists of a coupled system for the fluid mean velocity $u(x,t)$ on the one hand, and for the particle energy distribution function $g(x,\xi,t)$ on the other hand, where x and t denote position and time and where $\xi = \frac{1}{2}|v-u(x,t)|^2 \in [0,\infty)$ is the kinetic energy of a particle with velocity v in the fluid rest frame. This system of equations is written in dimension d ($d=1,2,3$) as

$$\begin{cases} \dfrac{\partial g}{\partial t} + u\cdot\nabla_x g - \dfrac{2}{d}\xi\dfrac{\partial g}{\partial \xi}(\nabla_x\cdot u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \nabla_x(\rho u u) + \dfrac{2}{d}\nabla_x W = 0, \end{cases} \tag{1.1}$$

where ρ and W are the fluid number and energy densities, related to g through

$$\rho = \int_0^\infty g\,\mathrm{d}\nu(\xi), \qquad W = \int_0^\infty \xi g\,\mathrm{d}\nu(\xi), \tag{1.2}$$

with $\mathrm{d}\nu(\xi) = |\mathbb{S}^{d-1}||2\xi|^{(d-2)/2}\,\mathrm{d}\xi$ and where $|\mathbb{S}^{d-1}|$ is the Lebesgue measure of the unit sphere in $\mathbb{R}^d$. The particle mass is set to 1 for simplicity. By integrating (1.1) with respect to ξ, a closed system of equations for the number and energy densities ρ and W and the mean velocity u is obtained, which turns out to be identical to the usual compressible Euler equations (see [5,6]). In this sense, we claim that the system (1.1) extends the classical compressible Euler model of fluid dynamics.

It was shown in [5,6] that this model can be formally derived from a hydrodynamic limit ($\varepsilon \to 0$) of the kinetic equation

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = \frac{1}{\varepsilon}Q(f), \tag{1.3}$$

where $f = f(x,v,t)$ is the kinetic distribution function and ε is the (supposedly small) Knudsen number. The collision operator $Q(f)$ describes the isotropization of the particle distribution function about the fluid mean velocity. This operator is written as

$$Q(f) = \frac{1}{\tau}(P_{u_f} f - f), \tag{1.4}$$

with

$$P_{u_f} f(x,v,t) = \frac{1}{4\pi}\int_{\mathbb{S}^{d-1}} f\big(x, u_f + |v-u_f|\omega, t\big)\,\mathrm{d}\omega, \tag{1.5}$$

$$u_f(x,t) = \left(\int v f(x,v,t)\,\mathrm{d}v\right)\left(\int f(x,v,t)\,\mathrm{d}v\right)^{-1}, \tag{1.6}$$

where u_f is the average velocity of f and $\tau = \tau(x,t)$ is the mean collision time. The collision model (1.3)–(1.6) appears in space plasma physics as a simplified description of wave–particle interactions and has proved to be useful in cosmic ray modeling (see, for example, [7–10]). The existence of solutions to the whole space initial value problem associated with (1.3) was proved in [11]. An existence theorem under milder assumptions, allowing for the occurrence of vacuum regions, was proven in [12]. Also, the incompressible limit of (1.3)–(1.6) towards the Navier–Stokes equation coupled with a transport–diffusion equation for the energy distribution function $g(x,\xi,t)$ has been computed rigorously in [13]. In [4], diffusive corrections to the macroscopic system (1.1) were derived from the kinetic model (1.3)–(1.6), in which τ was supposed to depend on the internal energy $\xi = \frac{1}{2}|v-u(x,t)|^2$ and not only on (x,t). In this case, the expression of u_f has to be modified in order to ensure momentum conservation: $\int_{\mathbb{R}^d} vQ(f)\,\mathrm{d}v = 0$. In particular, u_f is no longer the average velocity in the usual sense (1.6), but is linked with the distribution function f implicitly.

Nevertheless, the isotropy assumption ignores any statistically preferred direction. To overcome this (unrealistic) situation, in the present paper we consider a kinetic model of the form (1.3) with a collisional operator that relaxes the distribution function to an anisotropic equilibrium function. More precisely, given a velocity function $u(x,t)$ we define the set

$$\mathcal{N}_u = \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; f(v) = g\left(\frac{(v-u)_1^2}{2}, \frac{(v-u)_2^2}{2}, \ldots, \frac{(v-u)_d^2}{2}\right)\right\}, \tag{1.7}$$

where x_k denotes the k-th component of a vector x. To simplify the notation we set

$$\xi = \left(\frac{(v-u)_1^2}{2}, \frac{(v-u)_2^2}{2}, \ldots, \frac{(v-u)_d^2}{2}\right) \tag{1.8}$$

and then consider the following collision operator:

$$Q(f,u) = \frac{1}{\tau(x,\xi,t)}(\Pi_{\mathcal{N}_u} f - f), \tag{1.9}$$

where $\Pi_{\mathcal{N}_u}$ is the orthogonal projection onto $\mathcal{N}_u$ with respect to the usual scalar product in $L^2(\mathbb{R}^d)$. This collision operator relaxes the distribution function to an anisotropic equilibrium which is isotropic in each direction. We shall see that such a property provides a richer structure to the fluid limit of this kinetic model. The stress tensor, for instance, is still diagonal but is not proportional to the identity tensor. Furthermore, the obtained viscosity is a non-sparse matrix while it is simply a scalar for the isotropic model. As we shall also see, the first order fluid limit of the model (1.3) with the collision operator given by (1.9) is a coupled system of equations for the velocity $u(x,t)$ and the particle energy–vector distribution function $g(\xi_1, \xi_2, \dots, \xi_d)$:

$$\begin{cases} \dfrac{\partial g}{\partial t} + u \cdot \nabla_x g - 2\displaystyle\sum_{i=1}^{d} (\nabla_x u)_{ii} \xi_i \dfrac{\partial g}{\partial \xi_i} = 0, \\ \rho\left(\dfrac{\partial u}{\partial t} + (\nabla_x u) u\right)_k + 2\dfrac{\partial W_k}{\partial x_k} = 0, \quad \text{for } k = 1, 2, \dots, d, \end{cases} \tag{1.10}$$

where ρ and W_k are the fluid number density and the energy density in the k-th direction, related to g through

$$\rho = \int_{\mathbb{R}_+^d} g \, \mathrm{d}v(\xi), \qquad W_k = \int_{\mathbb{R}_+^d} \xi_k g \, \mathrm{d}v(\xi), \tag{1.11}$$

with $\mathrm{d}v(\xi) = 2^{-d/2}(\xi_1 \xi_2, \dots, \xi_d)^{-1/2} \, \mathrm{d}\xi$ (see [5,14] for a first approach to these results in the isotropic case). We stress the fact that, throughout this paper, the Einstein summation convention is not used. Indeed, the term $\partial W_k / \partial x_k$ does not imply summation over k.

It is important to note that the system (1.10) again contains the classical Euler equations of compressible fluid dynamics. The Euler system is obtained by simply integrating the first equation of (1.10) against 1 and ξ_k and summing over k. This model is in some sense an exact kinetic formulation of the Euler equation. In fact, a significant feature is that the system of moments of g derived from the first equation of (1.10) is closed at any order. In [4], the authors argue why such a model can be viewed as an Euler model with multivalued internal energy. We shall not reproduce the discussion here, but simply note that now the internal energy in each direction may be different and $\rho^{-1} g \, \mathrm{d}v(\xi)$ can be viewed as the probability that the internal energy in the k-th direction lies in the interval $[\xi_k, \xi_k + \mathrm{d}\xi_k]$ for all $1 \leqslant k \leqslant d$. We refer to [4] and [14] for some more precise justifications of the 'multivalued internal energy' terminology.

In this paper we also give expressions of diffusive corrections that can be added to the model (1.10). To that purpose we use Chapman–Enskog expansions starting from the kinetic model (1.3) with the collision operator defined by (1.9).

Finally, we analyze the collisional kinetic model in a variable frame. The isotropization in the directions of the laboratory frame leads to the fact that the stress tensor is diagonal in this fixed system of coordinates, which is not realistic for a general fluid. In some situations (when the stress tensor is not a scalar), the eigenvectors of the stress tensor in a fluid depend on the position x and on the time t. To account for this important point, the isotropization axes in the kinetic model have to depend on position and time and one has to consider an orthogonal transformation $R(x,t)$ from the laboratory system of coordinates to the isotropization axes, which are the eigenvectors of the stress tensor. The so obtained collision operator provides supplementary terms in its fluid limit, as we shall see later. The introduction of an orthogonal transformation leads to the natural question of how to choose the rotation matrix $R(x,t)$. Two possible solutions are investigated: the first one is to write a material derivative for this rotation that obeys the material invariance principle. The second way is to determine $R(x,t)$ by imposing an additional constraint on the collision operator such as the conservation of the stress tensor:

$$\int_{\mathbb{R}^d} v_i v_j Q(f) \, \mathrm{d}v = 0, \quad i, j = 1, \dots, d.$$

Then we analyze the mathematical and physical consequences that result from the explicit consideration of the rotational motion within the dynamical equations of the system, as well as their departure from conventional hydrodynamics. The origins and relevance of rotational motion in anisotropic fluid theories were explored in detail in [15] from a perspective concerning rigid body systems.

The paper is structured as follows: in Section 2 we give an overview of the main properties of the collision kernel (1.9) and transform the kinetic problem to local coordinates in order to make the analysis simpler. Section 3 is devoted to the derivation of the macroscopic dynamics induced by the collision operator (1.9) to the first and second orders. In Section 4 we derive the moment system associated with the multivalued energy fluid equation for g to the second order. Finally, Section 5 concerns the analysis of the anisotropic model in a variable frame.

2. The collision operator and the kinetic equation

2.1. The collision operator

We consider the collision operator defined by (1.9) written in the following form:

$$Q(f,u) = \frac{1}{\tau(x,\xi,t)} L_u f, \tag{2.1}$$

with

$$L_u f(v) = \Pi_{\mathcal{N}_u} f(v) - f(v). \tag{2.2}$$

The function $f = f(x,v,t)$ is the particle distribution function, depending on the position vector $x \in \mathbb{R}^d$, the velocity $v \in \mathbb{R}^d$ and the time $t > 0$. We recall that $\Pi_{\mathcal{N}_u}$ is the L^2-orthogonal projection onto the set $\mathcal{N}_u$ (defined by (1.7)) and that ξ is the vector expressed by (1.8), which represents the particle kinetic energy in all the coordinate directions and is measured in a reference frame moving with velocity u. We shall refer to ξ as the 'relative kinetic energy-vector'. On the other hand, the vector $u(x,t) = u_f(x,t)$ is some sort of average velocity of f determined by the requirement that Q is momentum-preserving. To make it explicit, we first need the following

Proposition 2.1. *The following assertions hold true*:

(i) *We have*

$$\Pi_{\mathcal{N}_u} Q(f,u) = 0, \tag{2.3}$$

or equivalently

$$\int_{\mathbb{R}^d} \phi\left(\frac{(v-u)_1^2}{2}, \frac{(v-u)_2^2}{2}, \ldots, \frac{(v-u)_d^2}{2}\right) Q(f,u)(v)\,\mathrm{d}v = 0, \tag{2.4}$$

for all functions $\phi(\xi)$ with $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$ and $\xi_k > 0$. In particular, $Q(f,u)$ preserves the density and the average fluid energy in the frame moving with velocity u, which is expressed by

$$\int_{\mathbb{R}^d} Q(f,u)(v)\,\mathrm{d}v = 0, \qquad \int_{\mathbb{R}^d} \frac{|v-u|^2}{2} Q(f,u)(v)\,\mathrm{d}v = 0. \tag{2.5}$$

(ii) *The null set of Q is the set $\bigcup_{u\in\mathbb{R}^d} \mathcal{N}_u$, where $\mathcal{N}_u$ is defined by* (1.7).

(iii) *The collision operator* (2.1), (2.2) *preserves momentum, i.e., it satisfies*

$$\int_{\mathbb{R}^d} v Q(f,u)(v)\,\mathrm{d}v = 0, \tag{2.6}$$

if and only if the velocity u satisfies

$$\int_{\mathbb{R}^d} (v-u)\tau^{-1}(\xi) f(v)\,\mathrm{d}v = 0, \tag{2.7}$$

with

$$\xi = (\xi_1, \xi_2, \ldots, \xi_d) = \left(\frac{(v-u)_1^2}{2}, \frac{(v-u)_2^2}{2}, \ldots, \frac{(v-u)_d^2}{2}\right).$$

In this case, the operator preserves the total fluid energy:

$$\int_{\mathbb{R}^d} \frac{|v|^2}{2} Q(f,u)(v)\,\mathrm{d}v = 0. \tag{2.8}$$

The proof is straightforward and can be deduced from [4].

In the remainder of the paper, we shall take $u = u_f$ satisfying (2.7) in the collision operator (2.1), (2.2). Note that Eq. (2.7) defines u_f implicitly (because of the dependence of τ^{-1} upon u_f). Also, the velocity u_f does not coincide with the usual fluid mean velocity $\bar{u} = \bar{u}_f$ defined by

$$\bar{u} = \frac{1}{\rho} \int_{\mathbb{R}^d} v f(v)\,\mathrm{d}v, \qquad \rho = \int_{\mathbb{R}^d} f(v)\,\mathrm{d}v. \tag{2.9}$$

However, in the particular case of a distribution function of the form $f(v) = f(\xi)$, both concepts of average velocities coincide: $u_f = \bar{u}_f$ (more precisely, $\bar{u}_f$ is a solution of (2.7)). In this paper, we shall not dwell on the problem of solving (2.7) and assume that there exists a unique 'physically admissible' velocity field solving (2.7).

2.2. The kinetic equation: approximate solutions and change to the local frame

We now consider the kinetic equation (1.3) with the collision operator defined by (2.1) and (2.2):

$$Tf^\varepsilon = \frac{1}{\varepsilon} Q\big(f^\varepsilon, u_{f^\varepsilon}\big), \qquad Tf \equiv \frac{\partial f}{\partial t} + v \cdot \nabla_x f, \tag{2.10}$$

and define an approximate solution of (2.10) at the order n to be a solution $\tilde{f}^\varepsilon$ of

$$T\tilde{f}^\varepsilon = \frac{1}{\varepsilon} Q\big(\tilde{f}^\varepsilon, u_{\tilde{f}^\varepsilon}\big) + \mathrm{O}\big(\varepsilon^n\big). \tag{2.11}$$

We show that Eq. (2.7) for the definition of u_f can be equivalently replaced by an equation involving the first moments of f. Indeed, Eq. (2.10) is equivalent to the following system, of unknowns f^ε and u^ε:

$$\begin{cases} Tf^\varepsilon = \dfrac{1}{\varepsilon} Q\big(f^\varepsilon, u^\varepsilon\big), \\ \displaystyle\int_{\mathbb{R}^d} \big(v - u^\varepsilon\big)\tau^{-1}\big(\xi, f^\varepsilon\big) f^\varepsilon(v)\,\mathrm{d}v = 0, \end{cases} \tag{2.12}$$

where ξ is linked to $v - u^\varepsilon$ through (1.8). Since the second equation of (2.12) is equivalent to the fact that Q is momentum-conservative, the system (2.12) is equivalent to

$$\begin{cases} Tf^\varepsilon = \dfrac{1}{\varepsilon} Q\big(f^\varepsilon, u^\varepsilon\big), \\ \displaystyle\int_{\mathbb{R}^d} vTf^\varepsilon(v)\,\mathrm{d}v = 0. \end{cases} \tag{2.13}$$

Then, u^ε appears as the Lagrange multiplier of the momentum preservation constraint as expressed by the second equation of (2.13).

The same splitting can be performed for the order n approximate solutions. Indeed, $\tilde{f}^\varepsilon$ is an approximate solution at the order n, according to the definition (2.11), if and only if there exists $\tilde{u}^\varepsilon$ such that $(\tilde{f}^\varepsilon, \tilde{u}^\varepsilon)$ satisfies

$$\begin{cases} T\tilde{f}^\varepsilon = \dfrac{1}{\varepsilon} Q\big(\tilde{f}^\varepsilon, \tilde{u}^\varepsilon\big) + \mathrm{O}\big(\varepsilon^n\big), \\ \displaystyle\int_{\mathbb{R}^d} vT\tilde{f}^\varepsilon(v)\,\mathrm{d}v = \mathrm{O}\big(\varepsilon^n\big). \end{cases} \tag{2.14}$$

The 'only if' part is obvious with the choice $\tilde{u}^\varepsilon = u_{\tilde{f}^\varepsilon}$ by multiplying (2.11) by v, integrating with respect to v and using that

$$\int_{\mathbb{R}^d} vQ\big(\tilde{f}^\varepsilon, u_{\tilde{f}^\varepsilon}\big)\,\mathrm{d}v = 0. \tag{2.15}$$

Conversely, from (2.14) and (2.15) we have

$$\int_{\mathbb{R}^d} vQ\big(\tilde{f}^\varepsilon, \tilde{u}^\varepsilon\big)\,\mathrm{d}v - \int_{\mathbb{R}^d} vQ\big(\tilde{f}^\varepsilon, u_{\tilde{f}^\varepsilon}\big)\,\mathrm{d}v = \mathrm{O}\big(\varepsilon^{n+1}\big).$$

Thus, under the hypothesis that Eq. (2.7) has a unique branch of 'physically admissible' solutions and that this branch consists of regular solutions of the nonlinear equation (2.7), we formally get

$$\tilde{u}^\varepsilon = u_{\tilde{f}^\varepsilon} + \mathrm{O}\big(\varepsilon^{n+1}\big).$$

This shows that every solution $\tilde{f}^\varepsilon$ of (2.14) is a solution to (2.11).

We now transform the approximate solutions defined by (2.14) by evaluating the kinetic velocities in the frame moving with velocity $\tilde{u}^\varepsilon$. We introduce the following notations:

$$p = v - \tilde{u}^\varepsilon, \qquad \widetilde{F}^\varepsilon(p) = \tilde{f}^\varepsilon(v), \qquad \xi = \left(\frac{p_1^2}{2}, \frac{p_2^2}{2}, \ldots, \frac{p_d^2}{2}\right) \in \mathbb{R}_+^d. \tag{2.16}$$

For simplicity, we shall omit the superscript ε, the tildes and the arguments t and x of the functions τ and f whenever the context is clear. We still denote by f and u the solutions to (2.14) and by F the corresponding function obtained after performing the change of variables (2.16). In terms of $F = F(p)$ we have

$$Tf = \frac{\partial F}{\partial t} + u\cdot\nabla_x F + p\cdot\nabla_x F - \left(\frac{\partial u}{\partial t} + (\nabla_x u)u\right)\cdot\nabla_p F - \big((\nabla_x u)p\big)\cdot\nabla_p F,$$

where $\nabla_x u$ denotes the matrix whose coefficients are $(\nabla_x u)_{ij} = \partial u_i/\partial x_j$. Then, the first equation of (2.14) becomes

$$\mathcal{A}(F,u) = \frac{1}{\varepsilon\tau(\xi,F)} LF + \mathrm{O}\big(\varepsilon^n\big), \tag{2.17}$$

where L is the operator L_0 given by (2.2) for $u = 0$:

$$LF(p) = \Pi F(p) - F(p), \tag{2.18}$$

with Π being the L^2-orthogonal projection onto

$$\mathcal{N} = \left\{ F \in L^2\big(\mathbb{R}^d\big) \;\middle|\; F(p) = g(\xi) \text{ with } \xi = \left(\frac{p_1^2}{2}, \frac{p_2^2}{2}, \ldots, \frac{p_d^2}{2}\right)\right\}, \tag{2.19}$$

and where the operator $\mathcal{A}$ is given by

$$\begin{cases} \mathcal{A}(F,u) = T_u F + p\cdot\nabla_x F - C_u\cdot\nabla_p F - \big((\nabla_x u)p\big)\cdot\nabla_p F,\\ T_u F = \dfrac{\partial F}{\partial t} + u\cdot\nabla_x F,\\ C_u = \dfrac{\partial u}{\partial t} + (\nabla_x u)u. \end{cases} \tag{2.20}$$

Now we recall that u and f are linked by the second equation of (2.14), which can be rewritten in terms of F as

$$\int_{\mathbb{R}^d} p\mathcal{A}(F,u)(p)\,\mathrm{d}p = \mathrm{O}\big(\varepsilon^n\big). \tag{2.21}$$

Using the expression of $\mathcal{A}$, we obtain by a simple integration

$$\int_{\mathbb{R}^d} p\mathcal{A}(F,u)(p)\,\mathrm{d}p = \mathcal{B}(F,u),$$

with

$$\mathcal{B}(F,u) = C_u \int_{\mathbb{R}^d} F\,\mathrm{d}p + \nabla_x\cdot\left(\int_{\mathbb{R}^d} p\otimes pF\,\mathrm{d}p\right) + \left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \nabla_x u + (\nabla_x\cdot u)I\right)\left(\int_{\mathbb{R}^d} pF\,\mathrm{d}p\right). \tag{2.22}$$

Finally, the problem (2.14) is equivalent to the following system of equations:

$$\begin{cases} \tau(\xi,F)\mathcal{A}(F,u) = \dfrac{1}{\varepsilon} LF + \mathrm{O}\big(\varepsilon^n\big),\\ \mathcal{B}(F,u) = \mathrm{O}\big(\varepsilon^n\big), \end{cases} \tag{2.23}$$

with $\mathcal{A}(F,u)$ given by (2.20) and $\mathcal{B}(F,u)$ by (2.22).

In (2.23), the collision operator L is linear and independent of u. L is clearly a self-adjoint operator (in L^2) and its null-space is simply the space $\mathcal{N}$ given by (2.19). Note also that the implicit character of Eq. (2.7) is now concentrated in the second equation of (2.23): given F, the corresponding u is simply a solution of an equation whose coefficients are averages of the distribution function F with respect to p.

3. Approximate macroscopic models to the first and second orders

3.1. The multivalued energy fluid model to the first order

In this section we check that the system (1.10) is indeed the first order approximate model associated with (2.10), for which we shall use the formulation (2.23). The Chapman–Enskog expansion at a kinetic scale is a classical asymptotic technique which bridges the gap between the Boltzmann-type equation modeling kinetic motion and the corresponding macroscopic fluid equations. For a summarized presentation of the Chapman–Enskog procedure and related bibliography we refer the readers to [4]. According to the Chapman–Enskog method, we assume that the first order approximate solution F to (2.23) has the form $F = F_0 + \varepsilon F_1$ and insert this expression into the system (2.23):

$$\begin{cases} \tau(\xi, F_0 + \varepsilon F_1)\big[\mathcal{A}(F_0, u) + \varepsilon \mathcal{A}(F_1, u)\big] = \dfrac{1}{\varepsilon} L F_0 + L F_1 + \mathrm{O}(\varepsilon), \\ \mathcal{B}(F_0, u) + \varepsilon \mathcal{B}(F_1, u) = \mathrm{O}(\varepsilon). \end{cases} \tag{3.1}$$

Now, identifying terms of the same order in ε and removing the terms of order ε we obtain

$$\begin{cases} L F_0 = 0, \\ \tau(\xi, F_0)\mathcal{A}(F_0, u) = L F_1, \\ \mathcal{B}(F_0, u) = 0. \end{cases} \tag{3.2}$$

The first equation of (3.2) implies that F_0 is a function of $\xi = (p_1^2/2, p_2^2/2, \ldots, p_d^2/2)$ only: $F_0(p) = g(\xi)$. The second equation of (3.2) admits a solution F_1 if and only if $\Pi \mathcal{A}(F_0, u) = 0$. We have

$$\mathcal{A}(F_0, u) = \mathcal{A}(g, u) = T_u g + \sum_{i=1}^{d} p_i \left(\frac{\partial g}{\partial x_i} - (C_u)_i \frac{\partial g}{\partial \xi_i} \right) - \sum_{i,j=1}^{d} (\nabla_x u)_{ij} p_i p_j \frac{\partial g}{\partial \xi_i}, \tag{3.3}$$

with T_u and C_u given in (2.20). To make $\Pi \mathcal{A}(g, u)$ and $\mathcal{B}(g, u)$ explicit, we use the following elementary lemma (the proof of which is omitted).

Lemma 3.1. *Let $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$ and ϕ be a function of*

$$\xi = \left(\frac{p_1^2}{2}, \frac{p_2^2}{2}, \ldots, \frac{p_d^2}{2} \right).$$

Also denote

$$A_{abc}^{def} = \delta_{ad}\delta_{be}\delta_{cf}, \qquad B_{abcd}^{ef} = \delta_{ab}\delta_{bc}\delta_{cd}\delta_{ef}, \qquad C_{abcdef} = \delta_{ab}\delta_{bc}\delta_{cd}\delta_{de}\delta_{ef},$$

where δ is the usual Kronecker symbol. Then, for all indices $i, j, k, l, m, n \in \{1, 2, \ldots, d\}$ we have

(i) $\Pi[p_i \phi(\xi)] = 0$.
(ii) $\Pi[p_i p_j \phi(\xi)] = 2\delta_{ij}\xi_i \phi(\xi)$.
(iii) $\Pi[p_i p_j p_k p_l \phi(\xi)] = 4[\delta_{ij}\delta_{kl}\xi_i\xi_k + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\xi_i\xi_j - 2\delta_{ij}\delta_{jk}\delta_{kl}\xi_i^2]\phi(\xi)$.
(iv)

$$\begin{aligned} \Pi\big[p_i p_j p_k p_l p_m p_n \phi(\xi)\big] = 8\big[& \big(A_{ijk}^{lmn} + A_{ijk}^{lnm} + A_{ijk}^{mln} + A_{ijk}^{mnl} + A_{ijk}^{nlm} + A_{ijk}^{nml}\big)\xi_i\xi_j\xi_k \\ & + \big(A_{ijl}^{kmn} + A_{ijl}^{knm} + A_{ijl}^{mkn} + A_{ijl}^{nkm}\big)\xi_i\xi_j\xi_l + \big(A_{ijm}^{kln} + A_{ijm}^{lkn}\big)\xi_i\xi_j\xi_m \\ & + \big(A_{ikl}^{jmn} + A_{ikl}^{jnm}\big)\xi_i\xi_k\xi_l + A_{ikm}^{jln}\xi_i\xi_k\xi_m \\ & - 2(1 - \delta_{ij})\big(B_{ikmn}^{jl} + B_{ilmn}^{jk} + B_{ikln}^{jm} + B_{iklm}^{jn}\big)\xi_i^2\xi_j \\ & - 2\big(B_{jlmn}^{ik} + B_{jkmn}^{il} + B_{jkln}^{im} + B_{jklm}^{in}\big)\xi_i\xi_j^2 \\ & - 2(1 - \delta_{ik})\big(B_{ijmn}^{kl} + B_{ijln}^{km} + B_{ijlm}^{kn}\big)\xi_i^2\xi_k - 2(1 - \delta_{il})\big(B_{ijkm}^{ln} + B_{ijkn}^{lm}\big)\xi_i^2\xi_l \\ & - 2(1 - \delta_{im})B_{ijkl}^{mn}\xi_i^2\xi_m - 2B_{klmn}^{ij}\xi_i\xi_k^2 - 4C_{ijklmn}\xi_i^3\big]\phi(\xi). \end{aligned}$$

Then, we easily get

$$\Pi\mathcal{A}(g,u) = T_u g - 2\sum_{i=1}^{d}(\nabla_x u)_{ii}\xi_i\frac{\partial g}{\partial \xi_i}, \qquad \mathcal{B}(g,u)_k = \rho(C_u)_k + 2\frac{\partial W_k}{\partial x_k}, \tag{3.4}$$

where ρ and W_k are given by (1.11). Thus, it follows that the system (3.2) is equivalent to the system (1.10) for g and u. At this level, the collision time $\tau(\xi,g)$ has no effect on the fluid model. In particular, at this order of approximation the discrepancy between t he fluid mean velocity $\bar{u}_f$ and the velocity $u = u_f$ is not detectable.

3.2. *The multivalued energy fluid model to the second order*

Now we seek an order two approximate solution to (2.13). We recall that this corresponds to a solution (F,u) to the system (2.23) for $n = 2$. As for the first order, we assume that F has the form

$$F = g(x,\xi,t) + \varepsilon F_1 + \varepsilon^2 F_2 \tag{3.5}$$

and insert this expression into the system (2.23):

$$\begin{cases} \tau\big(\xi, g+\varepsilon F_1+\varepsilon^2 F_2\big)\big\{\mathcal{A}(g,u)+\varepsilon\mathcal{A}(F_1,u)+\varepsilon^2\mathcal{A}(F_2,u)\big\} = LF_1+\varepsilon LF_2+\mathrm{O}\big(\varepsilon^2\big), \\ \mathcal{B}(g,u)+\varepsilon\mathcal{B}(F_1,u)+\varepsilon^2\mathcal{B}(F_2,u) = \mathrm{O}\big(\varepsilon^2\big). \end{cases} \tag{3.6}$$

The first equation implies (in particular) that $\Pi\mathcal{A}(g,u) = \mathrm{O}(\varepsilon)$. Thus, the first equation of (3.6) can be rewritten as

$$\begin{aligned} &\tau(\xi, g+\varepsilon F_1+\varepsilon^2 F_2)\left\{(I-\Pi)\mathcal{A}(g,u)+\varepsilon\left[\mathcal{A}(F_1,u)+\frac{1}{\varepsilon}\Pi\mathcal{A}(g,u)\right]+\varepsilon^2\mathcal{A}(F_2,u)\right\} \\ &\quad = LF_1+\varepsilon LF_2+\mathrm{O}\big(\varepsilon^2\big). \end{aligned} \tag{3.7}$$

We identify the terms of the same order in ε and remove the terms of order 2. We get

$$\begin{cases} \tau\big(\xi, g+\varepsilon F_1+\varepsilon^2 F_2\big)(I-\Pi)\mathcal{A}(g,u) = LF_1, \\ \tau\big(\xi, g+\varepsilon F_1+\varepsilon^2 F_2\big)\left[\mathcal{A}(F_1,u)+\dfrac{1}{\varepsilon}\Pi\mathcal{A}(g,u)\right] = LF_2, \\ \mathcal{B}(g,u)+\varepsilon\mathcal{B}(F_1,u) = 0. \end{cases} \tag{3.8}$$

Now we remark that the restriction of L to $N(L)^\perp$ is simply $-I$ and then its inverse L^{-1} is also equal to $-I$ when restricted to $N(L)^\perp$. If we seek $F_1 \in N(L)^\perp$, the solution of the first equation (3.8) is

$$F_1 = -\tau\big(\xi, g+\varepsilon F_1+\varepsilon^2 F_2\big)(I-\Pi)\mathcal{A}(g,u). \tag{3.9}$$

Then, the second equation of (3.8) has a non-empty set of solutions F_2 if and only if the projection Π of the left-hand side vanishes. Therefore, the last two equations of (3.8) are equivalent to

$$\begin{cases} \Pi\mathcal{A}(g,u)+\varepsilon\Pi\mathcal{A}(F_1,u) = 0, \\ \mathcal{B}(g,u)+\varepsilon\mathcal{B}(F_1,u) = 0. \end{cases} \tag{3.10}$$

Now, the expression of F_1 contains a dependence on $F = g+\varepsilon F_1+\varepsilon^2 F_2$ through the function τ. However, F_1 only appears in terms of order ε in (3.10). Therefore, with the same accuracy, we can replace $\tau(\xi,F)$ by $\tau(\xi,g)$ in (3.9) and get

$$F_1 = -\tau(\xi,g)(I-\Pi)\mathcal{A}(g,u). \tag{3.11}$$

The two equations (3.10) with F_1 given by (3.11) provide a sufficient condition on the pair (g,u) which makes F given by (3.5) an order two approximate solution of the kinetic model (2.10). Furthermore, any order two approximate solution of the form (3.5) with $F_1, F_2 \in N(L)^\perp$ is given by Eqs. (3.10), as the next result shows.

Proposition 3.2. *The pair* $(F = g+\varepsilon F_1+\varepsilon^2 F_2,\ u)$, *with* $\Pi F_1 = \Pi F_2 = 0$, *is an order two approximate solution to the kinetic equation* (2.13) *if and only if* (g,u) *is a solution of* (3.10).

Proof. We insert the expansion $F = g+\varepsilon F_1+\varepsilon^2 F_2$ into (2.23) and obtain (3.6). For the first equation of (3.6), we take the orthogonal projection onto $N(L)$ and find

$$\Pi\big[\mathcal{A}(g,u)+\varepsilon\mathcal{A}(F_1+\varepsilon F_2,u)\big] = \mathrm{O}\big(\varepsilon^2\big). \tag{3.12}$$

Then, the first equation of (3.6) is equivalent to

$$L(F_1+\varepsilon F_2)=\tau(\xi,F)\big[(I-\Pi)\mathcal{A}(g,u)+\varepsilon(I-\Pi)\mathcal{A}(F_1+\varepsilon F_2,u)\big]+\mathrm{O}\big(\varepsilon^2\big).$$

Since $F_1, F_2 \in N(L)^{\perp}$, we can now apply L^{-1} and obtain

$$F_1+\varepsilon F_2=\tau(\xi,F)L^{-1}\big[(I-\Pi)\mathcal{A}(g,u)+\varepsilon(I-\Pi)\mathcal{A}(F_1+\varepsilon F_2,u)\big]+\mathrm{O}\big(\varepsilon^2\big). \tag{3.13}$$

We insert the expression of $F_1+\varepsilon F_2$, given by (3.13), into (3.12) and get

$$\Pi\big[\mathcal{A}(g,u)+\varepsilon\mathcal{A}\big(\tau(\xi,F)L^{-1}\big[(I-\Pi)\mathcal{A}(g,u)\big],u\big)\big]=\mathrm{O}\big(\varepsilon^2\big).$$

With the second equation of (3.6) we finally obtain

$$\begin{cases}\Pi\big[\mathcal{A}(g,u)+\varepsilon\mathcal{A}\big(\tau(\xi,F)L^{-1}\big[(I-\Pi)\mathcal{A}(g,u)\big],u\big)\big]=\mathrm{O}\big(\varepsilon^2\big),\\ \mathcal{B}(g,u)+\varepsilon\mathcal{B}\big(\tau(\xi,F)L^{-1}\big[(I-\Pi)\mathcal{A}(g,u)\big],u\big)=\mathrm{O}\big(\varepsilon^2\big).\end{cases} \tag{3.14}$$

We can now replace $\tau(\xi,F)$ by $\tau(\xi,g)$ because we are only interested in terms of order less than two in ε. This leads to (3.10). Therefore, such an order two approximate solution is necessarily given by (3.10). □

Eq. (3.10) lead to the following model:

Proposition 3.3. *System* (3.10) *is equivalent to the following system*:

$$\begin{aligned}\frac{\partial g}{\partial t}&+u\cdot\nabla_x g-2\sum_{i=1}^{d}(\nabla_x u)_{ii}\xi_i\frac{\partial g}{\partial\xi_i}=2\varepsilon\sum_{i=1}^{d}\big[\xi_i\overline{\nabla}_i(\tau\overline{\nabla}_i g)-(C_u)_i\tau\overline{\nabla}_i g\big]\\&+2\varepsilon\sum_{i,j=1}^{d}(1-\delta_{ij})\big[(\nabla_x u)_{ij}^2\xi_j+(\nabla_x u)_{ij}(\nabla_x u)_{ji}\xi_i\big]\tau\frac{\partial g}{\partial\xi_i}\\&+4\varepsilon\sum_{i,j=1}^{d}(1-\delta_{ij})\xi_i\xi_j\left[(\nabla_x u)_{ij}^2\frac{\partial}{\partial\xi_i}\left(\tau\frac{\partial g}{\partial\xi_i}\right)+(\nabla_x u)_{ij}(\nabla_x u)_{ji}\frac{\partial}{\partial\xi_j}\left(\tau\frac{\partial g}{\partial\xi_i}\right)\right],\end{aligned} \tag{3.15}$$

$$\rho\left(\frac{\partial\bar u}{\partial t}+(\nabla_x\bar u)\bar u\right)_k+2\frac{\partial W_k}{\partial x_k}=\varepsilon\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\big[\mu_{ki}(\nabla_x\bar u)_{ik}+\mu_{ik}(\nabla_x\bar u)_{ki}\big], \tag{3.16}$$

where the symbol $\overline{\nabla}_i$ denotes the following oblique-derivative operator

$$\overline{\nabla}_i=\frac{\partial}{\partial x_i}-(C_u)_i\frac{\partial}{\partial\xi_i}, \tag{3.17}$$

and where u and $\bar u$ are linked by

$$\rho(\bar u-u)_k=-2\varepsilon\int_{\mathbb{R}^d}\xi_k\tau(\xi,g)\overline{\nabla}_k g\,\mathrm{d}p,\quad \textit{for } k=1,2,\dots,d. \tag{3.18}$$

Also, ρ and W_k are still defined by

$$\begin{pmatrix}\rho(x,t)\\ W_k(x,t)\end{pmatrix}=\int_{\mathbb{R}^d}\begin{pmatrix}1\\ \xi_k\end{pmatrix}g(x,\xi,t)\,\mathrm{d}p \tag{3.19}$$

and μ is the viscosity matrix whose coefficients are

$$\mu_{ij}=-4\int_{\mathbb{R}^d}(1-\delta_{ij})\xi_i\xi_j\tau(\xi,g)\frac{\partial g}{\partial\xi_i}\,\mathrm{d}p. \tag{3.20}$$

We recall that $\mathrm{d}p=2^{-d/2}(\xi_1\xi_2,\dots,\xi_d)^{-1/2}\,\mathrm{d}\xi$ *and* $\tau=\tau(\xi,g)$.

Before going to the proof of this proposition, we make some comments. The model (3.15)–(3.16) has still some similarities with the macroscopic system on $(g, \bar{u})$ derived in [4]. In the equation (3.15) for g, for instance, there are two diffusion terms: the first one is a diffusion operator in an oblique direction with respect to the variables (x_i, ξ_i) which is represented by the $\overline{\nabla}_i$ symbol. As in [4], this term would correspond to the heat flux in the moment system (4.1)–(4.3). The second diffusion term is a true diffusion with respect to the ξ_i variables. This term describes the effect of the viscosity on the distribution g and would correspond to the work of the viscosity forces in the moment system (4.1)–(4.3). However, the model (3.15), (3.16) is different from that of [4]. Indeed, the diffusion operators in both equations for g and u are more complex. In the equation of u, for instance, we observe that the viscous stress is not proportional to the strain-tensor $\sigma(u) = \nabla_x u + (\nabla_x u)^{\mathrm{T}} - \frac{2}{d}(\nabla_x \cdot u)I$ but depends on all the coefficients of the velocity tensor $\nabla_x u$. The viscosity stress and the velocity tensor are then related by a more complex relation that uses a tensor viscosity (with two indices) while this tensor is simply reduced to a scalar viscosity in the isotropic case. This is somehow similar to the order two closure relations in statistical models for turbulence (see, for example, [16–18]).

Proof. We already know the expression of $\Pi\mathcal{A}(g,u)$ from (3.4). Using (3.11) and (2.20), we have

$$F_1 = -\sum_{i=1}^{d} \tau p_i \overline{\nabla}_i g + \sum_{i,j=1}^{d} (1-\delta_{ij})(\nabla_x u)_{ij} p_i p_j \tau \frac{\partial g}{\partial \xi_i}. \tag{3.21}$$

Then, from the expression (2.20) of the operator $\mathcal{A}$ we obtain

$$\begin{aligned}
-\mathcal{A}(F_1,u) &= T_u\left(\sum_{i=1}^{d} \tau p_i \overline{\nabla}_i g\right) - T_u\left(\sum_{i,j=1}^{d} (1-\delta_{ij})(\nabla_x u)_{ij} p_i p_j \tau \frac{\partial g}{\partial \xi_i}\right) \\
&\quad + \sum_{i,j=1}^{d} p_i p_j \frac{\partial}{\partial x_j}(\tau \overline{\nabla}_i g) - \sum_{i,j=1}^{d} (C_u)_j p_i p_j \frac{\partial}{\partial \xi_j}(\tau \overline{\nabla}_i g) - \sum_{i=1}^{d} (C_u)_i \tau \overline{\nabla}_i g \\
&\quad - \sum_{i,j,k=1}^{d} (1-\delta_{ij}) p_i p_j p_k \frac{\partial}{\partial x_k}\left((\nabla_x u)_{ij} \tau \frac{\partial g}{\partial \xi_i}\right) + \sum_{i,j=1}^{d} (1-\delta_{ij})(C_u)_i p_j (\nabla_x u)_{ij} \tau \frac{\partial g}{\partial \xi_i} \\
&\quad + \sum_{i,j=1}^{d} (1-\delta_{ij})(C_u)_j p_i (\nabla_x u)_{ij} \tau \frac{\partial g}{\partial \xi_i} + \sum_{i,j,k=1}^{d} (1-\delta_{ij})(C_u)_k p_i p_j p_k (\nabla_x u)_{ij} \frac{\partial}{\partial \xi_k}\left(\tau \frac{\partial g}{\partial \xi_i}\right) \\
&\quad - \sum_{i,j=1}^{d} p_j (\nabla_x u)_{ij} \tau \overline{\nabla}_i g - \sum_{i,j,k=1}^{d} p_i p_j p_k (\nabla_x u)_{jk} \frac{\partial}{\partial \xi_j}(\tau \overline{\nabla}_i g) \\
&\quad + \sum_{i,j,k=1}^{d} (1-\delta_{ij}) p_j p_k (\nabla_x u)_{ij} (\nabla_x u)_{ik} \tau \frac{\partial g}{\partial \xi_i} + \sum_{i,j,k=1}^{d} (1-\delta_{ij}) p_i p_k (\nabla_x u)_{ij} (\nabla_x u)_{jk} \tau \frac{\partial g}{\partial \xi_i} \\
&\quad + \sum_{i,j,k,l=1}^{d} (1-\delta_{ij}) p_i p_j p_k p_l (\nabla_x u)_{ij} (\nabla_x u)_{kl} \frac{\partial}{\partial \xi_k}\left(\tau \frac{\partial g}{\partial \xi_i}\right).
\end{aligned} \tag{3.22}$$

Now, to take the projection of this expression we make use of Lemma 3.1(i)–(iii) and get

$$\begin{aligned}
-\Pi\mathcal{A}(F_1,u) &= \sum_{i=1}^{d}\left[2\xi_i \overline{\nabla}_i(\tau \overline{\nabla}_i g) - (C_u)_i \tau \overline{\nabla}_i g\right] + 2\sum_{i,j=1}^{d} (1-\delta_{ij})\left[(\nabla_x u)_{ij}^2 \xi_j + (\nabla_x u)_{ij} (\nabla_x u)_{ji} \xi_i\right] \tau \frac{\partial g}{\partial \xi_i} \\
&\quad + 4\sum_{i,j=1}^{d} (1-\delta_{ij}) \xi_i \xi_j \left[(\nabla_x u)_{ij}^2 \frac{\partial}{\partial \xi_i}\left(\tau \frac{\partial g}{\partial \xi_i}\right) + (\nabla_x u)_{ij} (\nabla_x u)_{ji} \frac{\partial}{\partial \xi_j}\left(\tau \frac{\partial g}{\partial \xi_i}\right)\right].
\end{aligned} \tag{3.23}$$

Inserting the above found expressions for $\Pi\mathcal{A}(g,u)$ (cf. (3.4)) and $\Pi\mathcal{A}(F_1,u)$ (cf. (3.23)) into the first equation of (3.10) leads to Eq. (3.15).

Now, to make the second equation of (3.10) explicit we need to compute $\mathcal{B}(g,u)$ (given by (3.4)) and $\mathcal{B}(F_1,u)$. With the expression of F_1 given by (3.21) and from the definition (2.22) of $\mathcal{B}$, we also have

$$\mathcal{B}(F_1,u)_k = \sum_{i,j,l=1}^{d} \frac{\partial}{\partial x_l}\left((\nabla_x u)_{ij} \int_{\mathbb{R}^d} (1-\delta_{ij}) p_i p_j p_k p_l \tau \frac{\partial g}{\partial \xi_i}\,\mathrm{d}p\right) - \left[\left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \nabla_x u + (\nabla_x\cdot u)I\right)\sum_{j=1}^{d}\int_{\mathbb{R}^d} p\otimes p_j \tau \overline{\nabla}_j g\,\mathrm{d}p\right]_k. \tag{3.24}$$

Nevertheless, using Lemma 3.1(iii) again the viscosity tensor can be simplified as

$$T_{ijkl} = -\int_{\mathbb{R}^d} (1-\delta_{ij}) p_i p_j p_k p_l \tau \frac{\partial g}{\partial \xi_i}\,\mathrm{d}p = \mu_{ij}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{3.25}$$

with μ defined by (3.20). Then,

$$\mathcal{B}(F_1,u)_k = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[\mu_{ki}(\nabla_x u)_{ik} + \mu_{ik}(\nabla_x u)_{ki}\right] - 2\left[\left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \nabla_x u + (\nabla_x\cdot u)I\right)\int_{\mathbb{R}^d} \xi\tau\overline{\nabla} g\,\mathrm{d}p\right]_k. \tag{3.26}$$

We now deduce that the second equation of (3.10) leads to

$$\rho\left(\frac{\partial u}{\partial t} + (\nabla_x u)u\right)_k + 2\frac{\partial W_k}{\partial x_k} = \varepsilon\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left[\mu_{ki}(\nabla_x u)_{ik} + \mu_{ik}(\nabla_x u)_{ki}\right] + 2\varepsilon\left[\left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \nabla_x u + (\nabla_x\cdot u)I\right)\int_{\mathbb{R}^d}\xi\tau\overline{\nabla} g\,\mathrm{d}p\right]_k. \tag{3.27}$$

Now, integrating Eq. (3.15) with respect to $\mathrm{d}p$ and introducing $\bar{u}$ according to (3.18), we obtain

$$\frac{\partial\rho}{\partial t} + \nabla_x\cdot(\rho u) = 2\varepsilon\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\int_{\mathbb{R}^d}\xi_i\tau\overline{\nabla}_i g\,\mathrm{d}p = -\nabla_x\cdot\left[\rho(\bar{u}-u)\right], \tag{3.28}$$

which is nothing else than the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\bar{u}) = 0 \tag{3.29}$$

for the velocity field $\bar{u}$. Then, repeatedly using the continuity equation (3.29) in the following computations, we have

$$\begin{aligned}
&\rho C_u + \left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \nabla_x u + (\nabla_x\cdot u)I\right)(\rho\bar{u} - \rho u)\\
&\quad = \left[\nabla_x\cdot(\rho\bar{u})\right]u + \left(\frac{\partial}{\partial t} + \nabla_x u\right)(\rho\bar{u}) + \left[u\cdot\nabla_x + (\nabla_x\cdot u)I\right](\rho\bar{u}-\rho u)\\
&\quad = \rho C_{\bar{u}} + \left[\nabla_x\cdot(\rho\bar{u})\right](u-\bar{u}) + (\nabla_x u - \nabla_x\bar{u})(\rho\bar{u}) + (\nabla_x\cdot u)(\rho\bar{u}-\rho u) + u\cdot\nabla_x(\rho\bar{u}-\rho u)\\
&\quad = \rho C_{\bar{u}} + (u-\bar{u})\cdot\nabla_x(\rho\bar{u}-\rho u) + \left[\nabla_x\cdot(u-\bar{u})\right](\rho\bar{u}-\rho u).
\end{aligned}$$

Since $u-\bar{u}$ is of the order of $\mathrm{O}(\varepsilon)$ (cf. (3.18)), we deduce that

$$\rho C_u + \left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \nabla_x u + (\nabla_x\cdot u)I\right)(\rho\bar{u}-\rho u) = \rho C_{\bar{u}} + \mathrm{O}(\varepsilon^2), \tag{3.30}$$

which implies that (3.27) is equivalent to Eq. (3.16) up to terms in ε^2. This achieves the proof of Proposition 3.3. □

Remark 1. Let $f(v) = F(v-u)$, with F given by (3.5). Then,

$$\int vf\,\mathrm{d}v = \int (v-u)f\,\mathrm{d}v + u\int f\,\mathrm{d}v = \int pF\,\mathrm{d}p + u\int F\,\mathrm{d}p = \rho u + \varepsilon\int pF_1\,\mathrm{d}p + \mathrm{O}(\varepsilon^2).$$

Furthermore, with (3.21) we have

$$\varepsilon\int p_k F_1\,\mathrm{d}p = -2\varepsilon\int_{\mathbb{R}^d}\xi_k\tau\overline{\nabla}_k g\,\mathrm{d}p = \rho(\bar{u}-u)_k.$$

Therefore,

$$\rho\bar{u} = \int vf\,\mathrm{d}v + \mathrm{O}(\varepsilon^2),$$

showing that $\bar{u}$ is the 'true' mean velocity of the distribution function up to terms of order ε^2. This remark explains why the continuity equation (3.29) and the momentum conservation (Eq. (3.16)) have a more natural expression once expressed in terms of $\bar{u}$.

Now we make some comments about the obtained model (3.15) and (3.16). First we point out the fact that this model does not contain that obtained in [4] where the considered collision operator is a relaxation to an isotropic equilibrium. In other words, even if we assume here g to be isotropic, we do not recover the istropic model obtained in [4]. This is essentially due to the following remark: if Π_I and Π_A denote the projections onto the spaces of isotropic and anisotropic functions respectively, then we have: $\Pi_I \neq \Pi_I \Pi_A$. Consider now the question about the beahavior of the diffusion operator in Eq. (3.16). We shall state a necessary and sufficient condition on the relaxation time τ for the diffusion operator to be negative. This is an important well-posedness condition for the model.

Lemma 3.4. *Let $\mathcal{D}$ be the linear operator acting on the velocity fields $\bar{u}(t,x)$ according to the following relation*

$$(\mathcal{D}\bar{u})_i = \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\bigl[\mu_{ji}(\nabla_x\bar{u})_{ij} + \mu_{ij}(\nabla_x\bar{u})_{ji}\bigr],$$

for $i = 1, 2, \ldots, d$. The viscosity matrix μ depends on the positive functions τ and g according to (3.20)*. Then, the operator $\mathcal{D}$ is negative on $(L^2(\mathbb{R}^d, \mathrm{d}x))^d$ if and only if*

$$\mu \text{ is a symmetric matrix, and } \mu_{ij} \geqslant 0, \text{ for all } i,\ j = 1, 2, \ldots, d.$$

Furthermore, the matrix μ is symmetric for all positive distribution function g if and only if the function τ has the form

$$\tau(\xi_1, \xi_2, \ldots, \xi_d) = (\xi_1\xi_2,\ldots,\xi_d)^{-1/2}\Upsilon(\xi_1 + \xi_2 + \cdots \xi_d), \tag{3.31}$$

where Υ is an arbitary positive and increasing function on $\mathbb{R}_+$.

Proof. We compute the scalar product of $\mathcal{D}\bar{u}$ by $\bar{u}$ in $(L^2(\mathbb{R}^d, \mathrm{d}x))^d$, we have

$$\begin{aligned}\langle\mathcal{D}\bar{u}, \bar{u}\rangle &= -\int_{\mathbb{R}^d} \sum_{i=1}^{d}\sum_{j=1}^{d}\bigl(\mu_{ji}(\nabla_x\bar{u})_{ij}^2 + \mu_{ij}(\nabla_x\bar{u})_{ji}(\nabla_x\bar{u})_{ij}\bigr)\,\mathrm{d}x,\\ &= -\frac{1}{2}\int_{\mathbb{R}^d} \sum_{i=1}^{d}\sum_{j=1}^{d}\bigl(\mu_{ji}(\nabla_x\bar{u})_{ij}^2 + \mu_{ij}(\nabla_x\bar{u})_{ji}^2 + (\mu_{ij}+\mu_{ji})(\nabla_x\bar{u})_{ji}(\nabla_x\bar{u})_{ij}\bigr)\,\mathrm{d}x.\end{aligned}$$

This quantity is negative for any field $(\nabla_x\bar{u})$ if and only if each term of the sum is also negative. This is equivalent to saying that the following quadratic form on $\mathbb{R}^2$

$$\mu_{ji}X^2 + \mu_{ij}Y^2 + (\mu_{ij}+\mu_{ji})XY,$$

is positive. It is an easy matter to check that this is also equivalent to

$$\forall i, j = 1, 2, \ldots, d, \quad \mu_{ij} \geqslant 0, \quad \text{and} \quad \begin{vmatrix} \mu_{ij} & \frac{1}{2}(\mu_{ij}+\mu_{ji}) \\ \frac{1}{2}(\mu_{ij}+\mu_{ji}) & \mu_{ji}\end{vmatrix} = -\frac{1}{4}(\mu_{ij}-\mu_{ji})^2 \geqslant 0,$$

which means that μ must be a symmetric matrix with positive coefficients.

Now, after an intergration by parts, the expression (3.20) of the viscosity μ can be written as

$$\mu_{ij} = 2(1-\delta_{ij})\int_{\mathbb{R}^d}\left(2\xi_i\xi_j\frac{\partial\tau}{\partial\xi_i} + \xi_j\tau\right)g\,\mathrm{d}p.$$

This matrix is symmetric for all g if and only if

$$2\xi_i\xi_j\frac{\partial\tau}{\partial\xi_i} + \xi_j\tau = 2\xi_i\xi_j\frac{\partial\tau}{\partial\xi_j} + \xi_i\tau, \quad \forall i, j = 1, 2, \ldots, d,$$

which is equivalent to

$$\left(\frac{\partial}{\partial \xi_i}-\frac{\partial}{\partial \xi_j}\right)\left(\log\tau+\frac{1}{2}\sum_{k=1}^{d}\log\xi_k\right)=0,\quad \forall i,\, j=1,2,\dots,d.$$

This means that τ must be of the form (3.31). Now using this expression, one can see that the coefficients μ_{ij} are all positive for any distribution function g, if and only if Υ is a positive and increasing function. □

4. The moment system

We multiply Eq. (3.15) for g successively by 1 and ξ_k, with $k=1,2,\dots,d$, and integrate with respect to $\mathrm{d}p = 2^{-d/2}(\xi_1\xi_2,\dots,\xi_d)^{-1/2}\,\mathrm{d}\xi$. We have the following

Proposition 4.4. *The model* (3.15), (3.16) *of Proposition* 3.3 *implies the following non-closed system of equations on the quantities* $(\rho,\bar u, W_k)$:

$$\frac{\partial\rho}{\partial t}+\nabla_x\cdot(\rho\bar u)=0, \tag{4.1}$$

$$\rho\left(\frac{\partial\bar u}{\partial t}+(\nabla_x\bar u)\bar u\right)_k+2\frac{\partial W_k}{\partial x_k}=\varepsilon\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left[\mu_{ik}(\nabla_x\bar u)_{ki}+\mu_{ki}(\nabla_x\bar u)_{ik}\right], \tag{4.2}$$

$$\frac{\partial W_k}{\partial t}+\nabla_x\cdot(\bar u W_k)+2(\nabla_x\bar u)_{kk}W_k=\varepsilon\sum_{i=1}^{d}\left[\mu_{ik}(\nabla_x\bar u)_{ki}^2+\mu_{ki}(\nabla_x\bar u)_{ik}(\nabla_x\bar u)_{ki}\right]+\varepsilon\nabla_x\cdot q_k, \tag{4.3}$$

where μ is the viscosity matrix related to g by (3.20) *and q_k (for $k=1,2,\dots,d$) is the k-th heat flux vector whose components are expressed by*

$$(q_k)_i=2\int_{\mathbb{R}^d}\xi_i\xi_k\tau\overline{\nabla}_i g\,\mathrm{d}p-2(1+2\delta_{ik})\frac{W_k}{\rho}\int_{\mathbb{R}^d}\xi_i\tau\overline{\nabla}_i g\,\mathrm{d}p. \tag{4.4}$$

Proof. Eqs. (4.1) and (4.2) are already known. To derive (4.3), (4.4) we first multiply Eq. (3.15) for g times ξ_k and integrate with respect to $\mathrm{d}p$. We have

$$\begin{aligned}&\frac{1}{\varepsilon}\left(\frac{\partial W_k}{\partial t}+\nabla_x\cdot(uW_k)+2(\nabla_x u)_{kk}W_k\right)\\&\quad=2\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(\int_{\mathbb{R}^d}\xi_i\xi_k\tau\overline{\nabla}_i g\,\mathrm{d}p\right)+2(C_u)_k\int_{\mathbb{R}^d}\xi_k\tau\overline{\nabla}_k g\,\mathrm{d}p\\&\qquad-4\sum_{i=1}^{d}(1-\delta_{ik})(\nabla_x u)_{ki}^2\int_{\mathbb{R}^d}\xi_i\xi_k\tau\frac{\partial g}{\partial\xi_k}\,\mathrm{d}p-4\sum_{i=1}^{d}(1-\delta_{ik})(\nabla_x u)_{ik}(\nabla_x u)_{ki}\int_{\mathbb{R}^d}\xi_i\xi_k\tau\frac{\partial g}{\partial\xi_i}\,\mathrm{d}p.\end{aligned}$$

Now, using the first order approximate macroscopic equation

$$\rho(C_u)_k+2\frac{\partial W_k}{\partial x_k}=0$$

and the relation (3.18) between u and $\bar u$ implies that

$$\begin{aligned}2(C_u)_k\int_{\mathbb{R}^d}\xi_k\tau\overline{\nabla}_k g\,\mathrm{d}p&=-\frac{2}{\varepsilon}\frac{\partial}{\partial x_k}\left[W_k(u-\bar u_k)\right]+\frac{2}{\varepsilon}\left[\nabla_x(u-\bar u\right]_{kk}W_k\\&=-\frac{1}{\varepsilon}\sum_{i=1}^{d}(1+2\delta_{ik})\frac{\partial}{\partial x_i}\left[(u-\bar u)_iW_k\right]+\frac{2}{\varepsilon}\left[\nabla_x(u-\bar u)\right]_{kk}W_k+\frac{1}{\varepsilon}\nabla_x\cdot\left[(u-\bar u)W_k\right],\end{aligned}$$

which concludes the proof. □

To close this section we make some comments. The vector q_k is the diffusion flux of the k-th component W_k of the internal energy vector $(W_1, W_2, \dots, W_d)$. We point out that q_k is not simply supported by the k-th axis of coordinates and comes from the oblique-diffusion terms (represented by the symbols $\overline{\nabla}_i$) in Eq. (3.15) for g. The term inside the brackets at the right-hand side of Eq. (4.3) is similar (but more complex) to the term $\mu\sigma(u) : \nabla_x u$ in the standard Navier–Stokes equations (see [4]), which measures the work of the viscosity force. This term is clearly associated with the viscous term in the equation for u (right-hand side of Eq. (4.2)) and comes from the diffusion terms in ξ_i in Eq. (3.15) for g. We finally point out that system (4.1)–(4.3) is not closed. Different strategies similar to those developed in [4] may be applied to close such a system. Here, we do not investigate these closure approaches but only refer to [4] for a detailed presentation.

5. The anisotropic model in a variable frame

It is often appropriate to introduce anisotropic viscosities to model turbulent diffusion. To this aim, we now consider a kinetic equation with a modified collision operator

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, u) = \frac{1}{\varepsilon} \frac{1}{\tau(x, \xi, t)} (\Pi_{\mathcal{N}_{u,R}} f - f) \tag{5.1}$$

which relaxes the particle distribution function f towards equilibrium functions of the form

$$g\left(\frac{(R(v-u))_1^2}{2}, \frac{(R(v-u))_2^2}{2}, \dots, \frac{(R(v-u))_d^2}{2}\right), \tag{5.2}$$

where $R = R(x, t)$ represents a direct orthogonal transformation (rotation) from the laboratory system of coordinates to the isotropization axes and where $\Pi_{\mathcal{N}_{u,R}}$ is the orthogonal projection onto the subspace of $L^2(\mathbb{R}^d)$ consisting of the functions (5.2).

Before going to the derivation of the corresponding macroscopic models ($\varepsilon \to 0$), we rewrite (5.1) in the following variables (similarly to the previous section)

$$p = R(v-u), \qquad f(v) = F(p). \tag{5.3}$$

The kinetic equation (5.1) becomes

$$\begin{cases} \tau \mathcal{A}(F, u, R) = \dfrac{1}{\varepsilon} L F(p) = \dfrac{1}{\varepsilon}\big[\Pi F(p) - F(p)\big], \\ \mathcal{B}(F, u, R) = \displaystyle\int_{\mathbb{R}^d} p \mathcal{A}(F, u, R)(p)\, \mathrm{d}p = 0 \end{cases} \tag{5.4}$$

with

$$\mathcal{A}(F, u, R) = T_u F + \big(R^{-1} p\big) \cdot \nabla_x F - \big[R C_u + \mathcal{U}^R p - \big(R^{-1} p\big) \cdot \nabla_x R \big(R^{-1} p\big)\big] \cdot \nabla_p F, \tag{5.5}$$

and

$$\mathcal{U}^R = (R \nabla_x u - T_u R) R^{-1}. \tag{5.6}$$

We recall that

$$T_u F = \frac{\partial F}{\partial t} + u \cdot \nabla_x F, \qquad C_u = \frac{\partial u}{\partial t} + (\nabla_x u) u$$

and that Π is the L^2 orthogonal projection onto

$$\mathcal{N} = \left\{ F \in L^2\big(\mathbb{R}^d\big) \;\middle|\; F(p) = g(\xi) \text{ with } \xi = \left(\frac{p_1^2}{2}, \dots, \frac{p_d^2}{2}\right)\right\}.$$

5.1. *Approximate macroscopic m odels to the first and second orders in a variable frame*

5.1.1. *The multivalued energy fluid model to the first order in a variable frame*

Here, our aim is to derive a first order approximation to (5.4) with respect to ε. Let $g = \Pi F$, we immediately get from (5.5)

$$\mathcal{A}(g, u, R) = T_u g + p \cdot (R \nabla_x) g - \sum_{i=1}^{d} (R C_u)_i p_i \frac{\partial g}{\partial \xi_i} - \sum_{i,j=1}^{d} p_i p_j \mathcal{U}_{ij}^R \frac{\partial g}{\partial \xi_i} + \sum_{i,j,k=1}^{d} \Omega_{ijk} p_i p_j p_k \frac{\partial g}{\partial \xi_i} \tag{5.7}$$

with

$$\Omega_{ijk} = \sum_{l,m=1}^{d} R_{jm} R_{kl} \frac{\partial R_{il}}{\partial x_m}. \tag{5.8}$$

Note that in writing the right-hand side of (5.7) we repeteadly used that $R^{-1} = R^{\mathrm{T}}$, R^{T} denoting the transpose matrix. Then, the results obtained in Section 3 can be used by just making the changes

$$C_u \longrightarrow RC_u, \quad \nabla_x u \longrightarrow \mathcal{U}^R, \quad \nabla_x \longrightarrow R\nabla_x \tag{5.9}$$

and considering the effects due to the new term

$$\Lambda^{\Omega}(g) = \sum_{i,j,k=1}^{d} \Omega_{ijk} p_i p_j p_k \frac{\partial g}{\partial \xi_i}, \tag{5.10}$$

as follows from a simple comparison with (3.3).

We now investigate the solvability conditions that make the Ansatz

$$F = g(\xi) + \varepsilon F_1 \tag{5.11}$$

an order one approximate solution of the kinetic model (2.13) with the collision kernel given by (5.1). Similarly to Section 3, a sufficient and necessary condition for F_1 to be a solution of $\tau(\xi, g)\mathcal{A}(g, u, R) = LF_1$ is

$$\Pi \mathcal{A}(g, u, R) = T_u g - 2\sum_{i=1}^{d} \mathcal{U}_{ii}^R \xi_i \frac{\partial g}{\partial \xi_i} = 0, \tag{5.12}$$

as deduced from a simple application of Lemma 3.1. Also, straightforward calculations lead to

$$\begin{aligned} \mathcal{B}(g, u, R)_k &= \int_{\mathbb{R}^d} p_k \mathcal{A}(g, u, R)\,\mathrm{d}p \\ &= \rho(RC_u)_k + 2\sum_{i=1}^{d} R_{ki} \frac{\partial W_k}{\partial x_i} - 2\sum_{i=1}^{d} (\Omega_{iik} + \Omega_{iki}) W_k - 2\sum_{i=1}^{d} \Omega_{kii} W_i, \end{aligned} \tag{5.13}$$

where the subscript k indicates the k-th component of the corresponding vector.

Then, it follows that the (first order) macroscopic equations associated with (5.11) are given by

$$\frac{\partial g}{\partial t} + u \cdot \nabla_x g - 2\sum_{i=1}^{d} \mathcal{U}_{ii}^R \xi_i \frac{\partial g}{\partial \xi_i} = 0, \tag{5.14}$$

$$\rho\left[R\frac{\partial u}{\partial t} + R(\nabla_x u)u\right]_k + 2\sum_{i=1}^{d} R_{ki} \frac{\partial W_k}{\partial x_i} - 2\sum_{i=1}^{d} (\Omega_{iik} + \Omega_{iki}) W_k - 2\sum_{i=1}^{d} \Omega_{kii} W_i = 0. \tag{5.15}$$

5.1.2. The multivalued energy fluid model to the second order in a variable frame

We now analyze the conditions under which

$$F = g(\xi) + \varepsilon F_1 + \varepsilon^2 F_2 \tag{5.16}$$

is an order two approximate solution of the kinetic model. To this aim, we first observe that the solution to the first equation of (3.8) is given by

$$F_1 = -\tau(\xi, g)(I - \Pi)\mathcal{A}(g, u, R) = -\sum_{i=1}^{d} \tau p_i \overline{\nabla}_i^R g + \sum_{i,j=1}^{d} (1 - \delta_{ij}) p_i p_j \mathcal{U}_{ij}^R \tau \frac{\partial g}{\partial \xi_i} - \tau \Lambda^{\Omega}(g), \tag{5.17}$$

where we denoted (cf. (3.17))

$$\overline{\nabla}_i^R g = \sum_{j=1}^{d} R_{ij} \frac{\partial g}{\partial x_j} - (RC_u)_i \frac{\partial g}{\partial \xi_i}$$

and used the expressions for $\Lambda^{\Omega}(g)$, $\mathcal{A}(g,u,R)$ and $\Pi\mathcal{A}(g,u,R)$ given in (5.10), (5.7) and (5.12), respectively.

Now, Proposition 3.2 applies again to ensure that F, defined by (5.16) with $\Pi F_1 = \Pi F_2 = 0$, is an order two approximate solution if and only if

$$\begin{cases} \Pi\mathcal{A}(g,u,R) + \varepsilon\Pi\mathcal{A}(F_1,u,R) = 0, \\ \mathcal{B}(g,u,R) + \varepsilon\mathcal{B}(F_1,u,R) = 0, \end{cases} \tag{5.18}$$

where $\mathcal{B}(F_1,u,R) = \int p\mathcal{A}(F_1,u,R)\,\mathrm{d}p$ and $\mathcal{B}(g,u,R)$ is given by (5.13). Hence, to derive the whole model for (g,u) it is enough to compute $\Pi\mathcal{A}(F_1,u,R)$ and $\mathcal{B}(F_1,u,R)$.

Lemma 5.3. *We have*

(i)
$$\begin{aligned} -\Pi\mathcal{A}(F_1,u,R) = {} & \sum_{i=1}^{d}\big[2\xi_i\overline{\nabla}_i^R\big(\tau\overline{\nabla}_i^R g\big) - (RC_u)_i\tau\overline{\nabla}_i^R g\big] + 2\sum_{i,j=1}^{d}(1-\delta_{ij})\big[\big(\mathcal{U}_{ij}^R\big)^2\xi_j + \mathcal{U}_{ij}^R\mathcal{U}_{ji}^R\xi_i\big]\tau\frac{\partial g}{\partial\xi_i} \\ & + 4\sum_{i,j=1}^{d}(1-\delta_{ij})\xi_i\xi_j\left[\big(\mathcal{U}_{ij}^R\big)^2\frac{\partial}{\partial\xi_i}\left(\tau\frac{\partial g}{\partial\xi_i}\right) + \mathcal{U}_{ij}^R\mathcal{U}_{ji}^R\frac{\partial}{\partial\xi_j}\left(\tau\frac{\partial g}{\partial\xi_i}\right)\right] \\ & - 2\sum_{i,j=1}^{d}\xi_i\big[(RC_u)_j\tau\overline{\nabla}_{iij}^{\Omega}g - \Omega_{jii}\tau\overline{\nabla}_j^R g\big] \\ & - 8\sum_{i=1}^{d}\xi_i^2\left[\overline{\nabla}_i^R\left(\Omega_{iii}\tau\frac{\partial g}{\partial\xi_i}\right) + \frac{\partial}{\partial\xi_i}\big(\Omega_{iii}\tau\overline{\nabla}_i^R g\big) + \sum_{j=1}^{d}\Omega_{jii}\tau\overline{\nabla}_{iij}^{\Omega}g\right] \\ & + 4\sum_{i,j=1}^{d}\xi_i\xi_j\left[\overline{\nabla}_{iij}^{\Omega}\big(\tau\overline{\nabla}_j^R g\big) + \overline{\nabla}_j^R\big(\tau\overline{\nabla}_{iij}^{\Omega}g\big) + \sum_{k=1}^{d}\big(\Omega_{kjj}\tau\overline{\nabla}_{iik}^{\Omega}g + \tau\overline{\nabla}_{kji}^{\Omega}g\big)\right] \\ & - 16\sum_{i,j=1}^{d}\xi_i^2\xi_j\left[\overline{\nabla}_{iij}^{\Omega}\big(\tau\overline{\nabla}_{iij}^{\Omega}g\big) + \frac{\partial}{\partial\xi_i}\big(\Omega_{iii}\tau\overline{\nabla}_{jji}^{\Omega}g\big) + \overline{\nabla}_{jji}^{\Omega}\left(\Omega_{iii}\tau\frac{\partial g}{\partial\xi_i}\right)\right] \\ & + 128\sum_{i=1}^{d}\xi_i^3\Omega_{iii}^2\frac{\partial}{\partial\xi_i}\left(\tau\frac{\partial g}{\partial\xi_i}\right) + 8\sum_{i,j,k=1}^{d}\xi_i\xi_j\xi_k\left[\overline{\overline{\nabla}}_{ijk}^{\Omega}\left(\tau\frac{\partial g}{\partial\xi_i}\right) + \overline{\nabla}_{jjk}^{\Omega}\big(\tau\overline{\nabla}_{iik}^{\Omega}g\big)\right], \end{aligned} \tag{5.19}$$

where we denoted

$$\overline{\nabla}_{ijk}^{\Omega}g = (\Omega_{ijk} + \Omega_{ikj})\frac{\partial g}{\partial\xi_i} + \Omega_{kji}\frac{\partial g}{\partial\xi_k}, \tag{5.20}$$

$$\overline{\overline{\nabla}}_{ijk}^{\Omega}g = (\Omega_{ijk} + \Omega_{ikj})\overline{\nabla}_{kji}^{\Omega}g. \tag{5.21}$$

(ii)
$$\begin{aligned} \mathcal{B}(F_1,u,R)_k = {} & -\sum_{i,j=1}^{d}\left[R_{ij}\frac{\partial}{\partial x_j} - (\Omega_{jij} + \Omega_{jji})\right]\big(\mu_{ki}\mathcal{U}_{ik}^R + \mu_{ik}\mathcal{U}_{ki}^R\big) \\ & + \sum_{i,j=1}^{d}\Omega_{kij}\big(\mu_{ji}\mathcal{U}_{ij}^R + \mu_{ij}\mathcal{U}_{ji}^R\big) - 2\left[\big(T_u + \mathcal{U}^R + \operatorname{tr}\big(\mathcal{U}^R\big)I\big)\int_{\mathbb{R}^d}\xi\tau\overline{\nabla}^R g\,\mathrm{d}p\right]_k \\ & - \left[\big(T_u + \mathcal{U}^R + \operatorname{tr}(\mathcal{U}^R)I\big)0\sum_{i,j,l=1}^{d}\Omega_{ijl}\int_{\mathbb{R}^d}p\otimes p_ip_jp_l\tau\frac{\partial g}{\partial\xi_i}\,\mathrm{d}p\right]_k, \end{aligned} \tag{5.22}$$

where μ_{ij} are the coefficients of the viscosity matrix given by (3.20) *and* $\operatorname{tr}(\mathcal{U}^R)$ *denotes the trace of* $\mathcal{U}^R = (R\nabla_x u - \partial R/\partial t - u\cdot\nabla_x R)R^{-1}$.

Proof. We first compute $\mathcal{A}(F_1,u,R)$. For that, it is convenient to consider the following splitting

$$\mathcal{A}(F_1,u,R) = \mathcal{A}_1(g,u,R) + \mathcal{A}_2(g,u,R),$$

where the first operator just retains the terms obtained by applying the transformations (5.9) to the results found for the case $R = I$ (cf. (3.22)) and the second one incorporates the additional terms due to $\Lambda^{\Omega}(g)$, given by (5.10). Indeed,

$$\mathcal{A}_1(g,u,R) = \mathcal{A}\big(F_1 + \tau\Lambda^{\Omega}(g), u, R\big) - \Lambda^{\Omega}\big(F_1 + \tau\Lambda^{\Omega}(g)\big), \tag{5.23}$$

$$\mathcal{A}_2(g,u,R) = \Lambda^{\Omega}\big(F_1 + \tau\Lambda^{\Omega}(g)\big) - \mathcal{A}\big(\tau\Lambda^{\Omega}(g), u, R\big). \tag{5.24}$$

Then, there only remains to calculate $\mathcal{A}_2(g,u,R)$. We have

$$\begin{aligned}
-\mathcal{A}_2(g,u,R) &= T_u\left(\sum_{i,j,k=1}^{d} \Omega_{ijk} p_i p_j p_k \tau \frac{\partial g}{\partial \xi_i}\right) + \sum_{i,j,k,l,m=1}^{d} p_i p_j p_k p_l R_{lm} \frac{\partial}{\partial x_m}\left(\Omega_{ijk}\tau\frac{\partial g}{\partial \xi_i}\right) \\
&\quad - \sum_{l=1}^{d}\left[(RC_u)_l + \sum_{m=1}^{d}\mathcal{U}^R_{lm} p_m\right] \sum_{i,j,k=1}^{d} \Omega_{ijk}\frac{\partial}{\partial p_l}\left(p_i p_j p_k \tau \frac{\partial g}{\partial \xi_i}\right) \\
&\quad + \sum_{l,m,n=1}^{d} \Omega_{lmn} p_m p_n \sum_{i,j,k=1}^{d} \Omega_{ijk}\frac{\partial}{\partial p_l}\left(p_i p_j p_k \tau \frac{\partial g}{\partial \xi_i}\right) \\
&\quad + \sum_{k,l,m=1}^{d} \Omega_{klm} p_l p_m \frac{\partial}{\partial p_k}\left(\sum_{i=1}^{d}\tau p_i \overline{\nabla}^R_i g - \sum_{i,j=1}^{d}(1-\delta_{ij}) p_i p_j \mathcal{U}^R_{ij}\tau\frac{\partial g}{\partial \xi_i}\right) \\
&= T_u\left(\sum_{i,j,k=1}^{d} p_i p_j p_k \Omega_{ijk}\tau\frac{\partial g}{\partial \xi_i}\right) + \sum_{i,j,k,l,m=1}^{d} p_i p_j p_k p_l R_{lm}\frac{\partial}{\partial x_m}\left(\Omega_{ijk}\tau\frac{\partial g}{\partial \xi_i}\right) \\
&\quad - \sum_{l=1}^{d}\left[(RC_u)_l + \sum_{m=1}^{d}\mathcal{U}^R_{lm} p_m - \sum_{m,n=1}^{d}\Omega_{lmn} p_m p_n\right]\sum_{i,j,k=1}^{d}\Omega_{ijk}\Bigg[(\delta_{il} p_j p_k + \delta_{jl} p_i p_k + \delta_{kl} p_i p_j) \\
&\quad \times \tau\frac{\partial g}{\partial \xi_i} + p_i p_j p_k p_l \frac{\partial}{\partial \xi_l}\left(\tau\frac{\partial g}{\partial \xi_i}\right)\Bigg] + \sum_{k,l,m=1}^{d}\Omega_{klm} p_l p_m\Bigg\{\sum_{i=1}^{d}\left[\delta_{ik}\tau\overline{\nabla}^R_i g + p_i p_k \frac{\partial}{\partial \xi_k}\big(\tau\overline{\nabla}^R_i g\big)\right] \\
&\quad - \sum_{i,j=1}^{d}(1-\delta_{ij})\mathcal{U}^R_{ij}\left[(\delta_{ik} p_j + \delta_{jk} p_i)\tau\frac{\partial g}{\partial \xi_i} + p_i p_j p_k \frac{\partial}{\partial \xi_k}\left(\tau\frac{\partial g}{\partial \xi_i}\right)\right]\Bigg\}.
\end{aligned} \tag{5.25}$$

Using again Lemma 3.1 repeteadly, elementary but lengthy computations lead to

$$\begin{aligned}
-\Pi\mathcal{A}_2(g,u,R) &= -2\sum_{i,j=1}^{d}\xi_i\big[(RC_u)_j\tau\overline{\nabla}^{\Omega}_{iij} g - \Omega_{jii}\tau\overline{\nabla}^R_j g\big] \\
&\quad - 8\sum_{i=1}^{d}\xi_i^2\left[\overline{\nabla}^R_i\left(\Omega_{iii}\tau\frac{\partial g}{\partial \xi_i}\right) + \frac{\partial}{\partial \xi_i}\big(\Omega_{iii}\tau\overline{\nabla}^R_i g\big) + \sum_{j=1}^{d}\Omega_{jii}\tau\overline{\nabla}^{\Omega}_{iij} g\right] \\
&\quad + 4\sum_{i,j=1}^{d}\xi_i\xi_j\left[\overline{\nabla}^{\Omega}_{iij}\big(\tau\overline{\nabla}^R_j g\big) + \overline{\nabla}^R_j\big(\tau\overline{\nabla}^{\Omega}_{iij} g\big) + \sum_{k=1}^{d}\big(\Omega_{kjj}\tau\overline{\nabla}^{\Omega}_{iik} g + \tau\overline{\overline{\nabla}}^{\Omega}_{kji} g\big)\right] \\
&\quad + 128\sum_{i=1}^{d}\xi_i^3\Omega_{iii}^2\frac{\partial}{\partial \xi_i}\left(\tau\frac{\partial g}{\partial \xi_i}\right) + 8\sum_{i,j,k=1}^{d}\xi_i\xi_j\xi_k\left[\overline{\overline{\nabla}}^{\Omega}_{ijk}\left(\tau\frac{\partial g}{\partial \xi_i}\right) + \overline{\nabla}^{\Omega}_{jjk}\big(\tau\overline{\nabla}^{\Omega}_{iik} g\big)\right] \\
&\quad - 16\sum_{i,j=1}^{d}\xi_i^2\xi_j\left[\overline{\nabla}^{\Omega}_{iij}\big(\tau\overline{\nabla}^{\Omega}_{iij} g\big) + \frac{\partial}{\partial \xi_i}\big(\Omega_{iii}\tau\overline{\nabla}^{\Omega}_{jji} g\big) + \overline{\nabla}^{\Omega}_{jji}\left(\Omega_{iii}\tau\frac{\partial g}{\partial \xi_i}\right)\right],
\end{aligned}$$

with the notations introduced in (5.20) and (5.21). This concludes the proof of (i).

We finally proceed to compute $\mathcal{B}(F_1,u,R)$. Integrating $p\mathcal{A}(F,u,R)$ with respect to p yields

$$\mathcal{B}(F,u,R) = (RC_u)\int_{\mathbb{R}^d} F\,\mathrm{d}p + (R\nabla_x)\cdot\int_{\mathbb{R}^d} p\otimes pF\,\mathrm{d}p$$
$$+\left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \mathcal{U}^R + \mathrm{tr}\big(\mathcal{U}^R\big)I\right)\int_{\mathbb{R}^d} pF\,\mathrm{d}p + \sum_{i,j,k=1}^{d}\Omega_{ijk}\int_{\mathbb{R}^d} p\otimes p_j p_k\frac{\partial F}{\partial p_i}\,\mathrm{d}p.$$

Then, when considering $F = F_1$ (given by (5.17)) we observe that the supplementary terms of $\mathcal{B}(F_1,u,R)$ (compared to (3.26)) due to rotational effects are those stemming from

$$\mathcal{B}\left(-\sum_{i,j,l=1}^{d}\Omega_{ijl}p_i p_j p_l\tau\frac{\partial g}{\partial\xi_i}, u, R\right)_k$$
$$+\sum_{i,j,l=1}^{d}\Omega_{ijl}\int_{\mathbb{R}^d} p_j p_k p_l\frac{\partial}{\partial p_i}\left(-\sum_{m=1}^{d}\tau p_m\overline{\nabla}_m^R g + \sum_{m,n=1}^{d}(1-\delta_{mn})p_m p_n\mathcal{U}_{mn}^R\tau\frac{\partial g}{\partial\xi_m}\right)\mathrm{d}p$$
$$=\sum_{i,j=1}^{d}\Omega_{kij}\big(\mu_{ji}\mathcal{U}_{ij}^R + \mu_{ij}\mathcal{U}_{ji}^R\big) + \sum_{i,j=1}^{d}(\Omega_{jji}+\Omega_{jij})\big(\mu_{ki}\mathcal{U}_{ik}^R + \mu_{ik}\mathcal{U}_{ki}^R\big)$$
$$-\left[\big(T_u + \mathcal{U}^R + \mathrm{tr}\big(\mathcal{U}^R\big)I\big)\sum_{i,j,l=1}^{d}\Omega_{ijl}\int_{\mathbb{R}^d} p\otimes p_i p_j p_l\tau\frac{\partial g}{\partial\xi_i}\,\mathrm{d}p\right]_k.$$

Now we are done with the proof. □

The above calculations lead to the following result.

Proposition 5.5. *If* (F,u) *is a solution to* (5.4)*, then* $(g = \Pi F, u)$ *satisfies the following system of equations up to terms of order* 2 *in* ε:

$$\frac{\partial g}{\partial t} + u\cdot\nabla_x g - 2\sum_{i=1}^{d}\mathcal{U}_{ii}^R\xi_i\frac{\partial g}{\partial\xi_i}$$
$$=\varepsilon\sum_{i=1}^{d}\big[2\xi_i\overline{\nabla}_i^R\big(\tau\overline{\nabla}_i^R g\big) - (RC_u)_i\tau\overline{\nabla}_i^R g\big] + 2\varepsilon\sum_{i,j=1}^{d}(1-\delta_{ij})\big[\big(\mathcal{U}_{ij}^R\big)^2\xi_j + \mathcal{U}_{ij}^R\mathcal{U}_{ji}^R\xi_i\big]\tau\frac{\partial g}{\partial\xi_i}$$
$$+4\varepsilon\sum_{i,j=1}^{d}(1-\delta_{ij})\xi_i\xi_j\left[\big(\mathcal{U}_{ij}^R\big)^2\frac{\partial}{\partial\xi_i}\left(\tau\frac{\partial g}{\partial\xi_i}\right) + \mathcal{U}_{ij}^R\mathcal{U}_{ji}^R\frac{\partial}{\partial\xi_j}\left(\tau\frac{\partial g}{\partial\xi_i}\right)\right]$$
$$-8\varepsilon\sum_{i=1}^{d}\xi_i^2\left[\overline{\nabla}_i^R\left(\Omega_{iii}\tau\frac{\partial g}{\partial\xi_i}\right) + \frac{\partial}{\partial\xi_i}\big(\Omega_{iii}\tau\overline{\nabla}_i^R g\big) + \sum_{j=1}^{d}\Omega_{jii}\tau\overline{\nabla}_{iij}^{\Omega} g\right]$$
$$-2\varepsilon\sum_{i,j=1}^{d}\xi_i\big[(RC_u)_j\tau\overline{\nabla}_{iij}^{\Omega} g - \Omega_{jii}\tau\overline{\nabla}_j^R g\big] + 4\varepsilon\sum_{i,j=1}^{d}\xi_i\xi_j\Bigg[\overline{\nabla}_{iij}^{\Omega}\big(\tau\overline{\nabla}_j^R g\big) + \overline{\nabla}_j^R\big(\tau\overline{\nabla}_{iij}^{\Omega} g\big)$$
$$+\sum_{k=1}^{d}\big(\Omega_{kjj}\tau\overline{\nabla}_{iik}^{\Omega} g + \tau\overline{\overline{\nabla}}_{kji}^{\Omega} g\big)\Bigg] + 128\varepsilon\sum_{i=1}^{d}\xi_i^3\Omega_{iii}^2\frac{\partial}{\partial\xi_i}\left(\tau\frac{\partial g}{\partial\xi_i}\right)$$
$$-16\varepsilon\sum_{i,j=1}^{d}\xi_i^2\xi_j\left[\overline{\nabla}_{iij}^{\Omega}\big(\tau\overline{\nabla}_{iij}^{\Omega} g\big) + \frac{\partial}{\partial\xi_i}\big(\Omega_{iii}\tau\overline{\nabla}_{jji}^{\Omega} g\big) + \overline{\nabla}_{jji}^{\Omega}\left(\Omega_{iii}\tau\frac{\partial g}{\partial\xi_i}\right)\right]$$
$$+8\varepsilon\sum_{i,j,k=1}^{d}\xi_i\xi_j\xi_k\left[\overline{\overline{\nabla}}_{ijk}^{\Omega}\left(\tau\frac{\partial g}{\partial\xi_i}\right) + \overline{\nabla}_{jjk}^{\Omega}\big(\tau\overline{\nabla}_{iik}^{\Omega} g\big)\right], \tag{5.26}$$

$$\rho\left[R\left(\frac{\partial \bar{u}}{\partial t}+(\nabla_x \bar{u})\bar{u}\right)\right]_k+2\sum_{i=1}^{d} R_{ki}\frac{\partial W_k}{\partial x_i}-2\sum_{i=1}^{d}(\Omega_{iik}+\Omega_{iki})W_k-2\sum_{i=1}^{d}\Omega_{kii}W_i$$
$$=\varepsilon\sum_{i,j=1}^{d}\left[R_{ij}\frac{\partial}{\partial x_j}-(\Omega_{jji}+\Omega_{jij})\right]\left(\mu_{ki}\overline{\mathcal{U}}^R_{ik}+\mu_{ik}\overline{\mathcal{U}}^R_{ki}\right)-\varepsilon\sum_{i,j=1}^{d}\Omega_{kij}\left(\mu_{ji}\overline{\mathcal{U}}^R_{ij}+\mu_{ij}\overline{\mathcal{U}}^R_{ji}\right), \tag{5.27}$$

where $\tau=\tau(\xi,g)$ and the symbols $\overline{\nabla}^R_i$, $\overline{\nabla}^\Omega_{ijk}$ and $\overline{\overline{\nabla}}^\Omega_{ijk}$ denote the following oblique–derivative operators

$$\overline{\nabla}^R_i g=\sum_{j=1}^{d} R_{ij}\frac{\partial g}{\partial x_j}-(RC_u)_i\frac{\partial g}{\partial \xi_i},$$
$$\overline{\nabla}^\Omega_{ijk} g=(\Omega_{ijk}+\Omega_{ikj})\frac{\partial g}{\partial \xi_i}+\Omega_{kji}\frac{\partial g}{\partial \xi_k},$$
$$\overline{\overline{\nabla}}^\Omega_{ijk} g=(\Omega_{ijk}+\Omega_{ikj})\overline{\nabla}^\Omega_{kji} g,$$

and where u and $\bar{u}$ are connected by

$$\rho\left[R(\bar{u}-u)\right]_k=2\varepsilon\left(\int_{\mathbb{R}^d}\xi_k\tau(\xi,g)\overline{\nabla}^R_k g\,\mathrm{d}p+2\sum_{j=1}^{d}\int_{\mathbb{R}^d}\xi_j\xi_k\tau(\xi,g)\overline{\nabla}^\Omega_{jjk} g\,\mathrm{d}p-4\Omega_{kkk}\int_{\mathbb{R}^d}\xi_k^2\tau(\xi,g)\frac{\partial g}{\partial \xi_k}\,\mathrm{d}p\right) \tag{5.28}$$

for $k=1,2,\ldots,d$. Here, $\mathrm{d}p=2^{-d/2}(\xi_1\xi_2,\ldots,\xi_d)^{-1/2}\,\mathrm{d}\xi$. Also, the density ρ and the internal energy W are still defined by the following moments of g

$$\begin{pmatrix}\rho(x,t)\\ W_k(x,t)\end{pmatrix}=\int_{\mathbb{R}^d}\begin{pmatrix}1\\ \xi_k\end{pmatrix}g(x,\xi,t)\,\mathrm{d}p,$$

and μ is the viscosity matrix with coefficients

$$\mu_{ij}=-4\int_{\mathbb{R}^d}(1-\delta_{ij})\xi_i\xi_j\tau(\xi,g)\frac{\partial g}{\partial \xi_i}\,\mathrm{d}p.$$

The proof is analogous to that of Proposition 3.3 by making use of Lemma 5.3. We just remark that $\bar{u}$ chosen as in (5.28) is again a 'true' mean velocity of the distribution function up to second order terms in ε. Indeed, integrating Eq. (5.26) for g against p yields

$$\frac{\partial \rho}{\partial t}+u\cdot\nabla_x\rho+\mathrm{tr}\left(\mathcal{U}^R\right)\rho=2\varepsilon\sum_{i,j=1}^{d}\left[R_{ij}\frac{\partial}{\partial x_j}-(\Omega_{jji}+\Omega_{jij})\right]\left(\int_{\mathbb{R}^d}\xi_i\tau(\xi,g)\overline{\nabla}^R_i g\,\mathrm{d}p\right.$$
$$\left.+2\Omega_{iii}\int_{\mathbb{R}^d}\xi_i^2\tau(\xi,g)\frac{\partial g}{\partial \xi_i}\,\mathrm{d}p+2\sum_{k=1}^{d}\int_{\mathbb{R}^d}(1-\delta_{ik})\xi_i\xi_k\tau(\xi,g)\overline{\nabla}^\Omega_{kki} g\,\mathrm{d}p\right).$$

Introducing now $\bar{u}$ according to (5.28) we find

$$\frac{\partial \rho}{\partial t}+u\cdot\nabla_x\rho+\mathrm{tr}\left(\mathcal{U}^R\right)\rho=\sum_{i,j=1}^{d}\left[R_{ij}\frac{\partial}{\partial x_j}-(\Omega_{jji}+\Omega_{jij})\right]\left(\rho\left[R(\bar{u}-u)\right]_i\right),$$

which eventually becomes the balance equation

$$\frac{\partial \rho}{\partial t}+\bar{u}\cdot\nabla_x\rho+\mathrm{tr}\left(\overline{\mathcal{U}}^R\right)\rho=0 \tag{5.29}$$

by just accounting for

$$\left[R_{ij}\frac{\partial}{\partial x_j}-(\Omega_{jji}+\Omega_{jij})\right]\left(\rho\left[R(\bar{u}-u)\right]_i\right)=(u-\bar{u})\cdot\nabla_x\rho+\mathrm{tr}\left[\left(R\nabla_x(u-\bar{u})-(u-\bar{u})\cdot\nabla_x R\right)R^{-1}\right]\rho.$$

Here, we denoted

$$\overline{\mathcal{U}}^R = \big(R\nabla_x \bar{u} - T_{\bar{u}} R\big)R^{-1}. \tag{5.30}$$

Then, we have

$$\begin{aligned}
&\rho(RC_u) + \left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \mathcal{U}^R + \operatorname{tr}\big(\mathcal{U}^R\big)I\right)\big[\rho R(\bar{u}-u)\big] \\
&\quad = \bar{u}\cdot\nabla_x \rho(Ru) + \left(\frac{\partial}{\partial t} + \mathcal{U}^R\right)\big[\rho(R\bar{u})\big] + \operatorname{tr}\big(\overline{\mathcal{U}}^R\big)I(\rho R u) + \operatorname{tr}\big(\mathcal{U}^R\big)I\big[\rho R(\bar{u}-u)\big] + R(u\cdot\nabla_x)\big[\rho(\bar{u}-u)\big] \\
&\quad = \rho(RC_{\bar{u}}) + \operatorname{tr}\big(\overline{\mathcal{U}}^R\big)I\big[\rho R(\bar{u}-u)\big] + R(u\cdot\nabla_x)\big[\rho(\bar{u}-u)\big] \\
&\qquad + \big[R\nabla_x(u-\bar{u})\big](\rho\bar{u}) + \big(u\cdot\nabla_x\rho + \operatorname{tr}\big(\overline{\mathcal{U}}^R\big)\rho I\big)\big[R(u-\bar{u})\big] \\
&\quad = \rho(RC_{\bar{u}}) + \operatorname{tr}\big(\mathcal{U}^R - \overline{\mathcal{U}}^R\big)I\big[\rho R(\bar{u}-u)\big] + R\big[(\bar{u}-u)\cdot\nabla_x\big]\big[\rho(u-\bar{u})\big],
\end{aligned}$$

where we have repeteadly used the continuity equation (5.29). Consequently, we deduce from (5.28) that

$$\rho(RC_u) + \left(\frac{\partial}{\partial t} + u\cdot\nabla_x + \mathcal{U}^R + \operatorname{tr}\big(\mathcal{U}^R\big)I\right)\rho R(\bar{u}-u) = \rho(RC_{\bar{u}}) + \mathrm{O}\big(\varepsilon^2\big),$$

which implies that Eq. (5.27) is satisfied up to terms in ε^2.

Finally, repeating the argument of Remark 1 we get

$$\int vf\,\mathrm{d}v = \int \big(R^{-1}p\big)F\,\mathrm{d}p + u\int F\,\mathrm{d}p = \rho u + \varepsilon\int \big(R^{-1}p\big)F_1\,\mathrm{d}p + \mathrm{O}\big(\varepsilon^2\big),$$

where $f(v) = F(v-u)$ with F given by (3.5). Now, inserting (5.17) into the above expression yields

$$\varepsilon\int \big(R^{-1}p\big)_l F_1\,\mathrm{d}p = -\varepsilon\int \big(R^{-1}p\big)_l\left(\sum_{i=1}^d \tau p_i \overline{\nabla}_i^R g + \sum_{i,j,k=1}^d \Omega_{ijk} p_i p_j p_k \tau\frac{\partial g}{\partial \xi_i}\right)\mathrm{d}p = \rho(\bar{u}-u)_l,$$

so that

$$\rho\bar{u} = \int vf\,\mathrm{d}v + \mathrm{O}\big(\varepsilon^2\big).$$

5.2. The moment system

In this section we derive the system of moment equations for the second order approximate model (5.26), (5.27). Multiplying Eq. (5.26) for g successively by 1 and ξ_k, with $k = 1, 2, \ldots, d$, and integrating against $\mathrm{d}p = 2^{-d/2}(\xi_1\xi_2,\ldots,\xi_d)^{-1/2}\,\mathrm{d}\xi$ yields the following

Proposition 5.6. *The model* (5.26), (5.27) *of Proposition* 5.5 *implies the following non-closed system of equations on the quantities* $(\rho, \bar{u}, W_k)$:

$$\frac{\partial\rho}{\partial t} + \bar{u}\cdot\nabla_x\rho + \operatorname{tr}\big(\overline{\mathcal{U}}^R\big)\rho = 0, \tag{5.31}$$

$$\begin{aligned}
&\rho\left[R\left(\frac{\partial\bar{u}}{\partial t} + (\nabla_x\bar{u})\bar{u}\right)\right]_k + 2\sum_{i=1}^d R_{ki}\frac{\partial W_k}{\partial x_i} - 2\sum_{i=1}^d(\Omega_{iik}+\Omega_{iki})W_k - 2\sum_{i=1}^d \Omega_{kii}W_i \\
&\quad = \varepsilon\sum_{i,j=1}^d\left[R_{ij}\frac{\partial}{\partial x_j} - (\Omega_{jji}+\Omega_{jij})\right]\big(\mu_{ki}\overline{\mathcal{U}}_{ik}^R + \mu_{ik}\overline{\mathcal{U}}_{ki}^R\big) - \varepsilon\sum_{i,j=1}^d \Omega_{kij}\big(\mu_{ji}\overline{\mathcal{U}}_{ij}^R + \mu_{ij}\overline{\mathcal{U}}_{ji}^R\big),
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
&\frac{\partial W_k}{\partial t} + \bar{u}\cdot\nabla_x W_k + \operatorname{tr}\big(\overline{\mathcal{U}}^R\big)W_k + 2\overline{\mathcal{U}}_{kk}^R W_k \\
&\quad = \varepsilon\sum_{i=1}^d\big[\mu_{ik}\big(\overline{\mathcal{U}}_{ki}^R\big)^2 + \mu_{ki}\overline{\mathcal{U}}_{ik}^R\overline{\mathcal{U}}_{ki}^R\big] - 8\varepsilon\sum_{i,j=1}^d\int_{\mathbb{R}^d}\xi_i\xi_j\xi_k\tau\overline{\overline{\nabla}}_{kji}^\Omega g\,\mathrm{d}p
\end{aligned}$$

$$+ 8\varepsilon\Omega_{kkk}\left(\int_{\mathbb{R}^d}\xi_k^2\tau\frac{\partial g}{\partial\xi_k}\,\mathrm{d}p - 16\Omega_{kkk}\int_{\mathbb{R}^d}\xi_k^3\tau\frac{\partial g}{\partial\xi_k}\,\mathrm{d}p + 2\sum_{l=1}^d\int_{\mathbb{R}^d}\xi_k^2\xi_l\tau\overline{\nabla}^{\Omega}_{llk}g\,\mathrm{d}p\right)$$

$$- 24\varepsilon\Omega_{kkk}\frac{W_k}{\rho}\left(\int_{\mathbb{R}^d}\xi_k\tau\overline{\nabla}^R_k g\,\mathrm{d}p - 4\Omega_{kkk}\int_{\mathbb{R}^d}\xi_k^2\tau\frac{\partial g}{\partial\xi_k}\,\mathrm{d}p + 2\sum_{l=1}^d\int_{\mathbb{R}^d}\xi_k\xi_l\tau\overline{\nabla}^{\Omega}_{llk}g\,\mathrm{d}p\right)$$

$$+ 16\varepsilon\sum_{i=1}^d\left[(\Omega_{kki}+\Omega_{kik})\int_{\mathbb{R}^d}\xi_i\xi_k^2\tau\overline{\nabla}^{\Omega}_{kki}g\,\mathrm{d}p + \Omega_{kii}\int_{\mathbb{R}^d}\xi_i^2\xi_k\tau\overline{\nabla}^{\Omega}_{iik}g\,\mathrm{d}p\right]$$

$$+ \varepsilon\sum_{i,j=1}^d\left[R_{ij}\frac{\partial}{\partial x_j} - (\Omega_{jji}+\Omega_{jij})\right](q_k)_i - 2\varepsilon\sum_{i=1}^d(\Omega_{kki}+\Omega_{kik})(q_k)_i - 2\varepsilon\sum_{i=1}^d\Omega_{kii}(q_i)_k, \tag{5.33}$$

where μ is the viscosity matrix related to g by (3.20) *and q_k (for $k = 1, 2, \dots, d$) is the k-th heat flux vector whose components are given by*

$$(q_k)_i = 2\int_{\mathbb{R}^d}\xi_i\xi_k\tau(\xi,g)\overline{\nabla}^R_i g\,\mathrm{d}p - 8\Omega_{iii}\int_{\mathbb{R}^d}\xi_i^2\xi_k\tau(\xi,g)\frac{\partial g}{\partial\xi_i}\,\mathrm{d}p + 4\sum_{l=1}^d\int_{\mathbb{R}^d}\xi_i\xi_k\xi_l\tau(\xi,g)\overline{\nabla}^{\Omega}_{lli}g\,\mathrm{d}p$$
$$- (1+2\delta_{ik})\frac{W_k}{\rho}\left(2\int_{\mathbb{R}^d}\xi_i\tau\overline{\nabla}^R_i g\,\mathrm{d}p - 8\Omega_{iii}\int_{\mathbb{R}^d}\xi_i^2\tau\frac{\partial g}{\partial\xi_i}\,\mathrm{d}p + 4\sum_{l=1}^d\int_{\mathbb{R}^d}\xi_i\xi_l\tau\overline{\nabla}^{\Omega}_{lli}g\,\mathrm{d}p\right). \tag{5.34}$$

Proof. Eqs. (5.31) and (5.32) were already established. We then multiply Eq. (5.26) for g times ξ_k and integrate with respect to $\mathrm{d}p$ to obtain

$$\frac{1}{\varepsilon}\left[\frac{\partial W_k}{\partial t} + \nabla_x\cdot(uW_k) + \operatorname{tr}(\overline{\mathcal{U}}^R)W_k + 2(\nabla_x u)_{kk}W_k\right] = \sum_{i=1}^d\left[\mu_{ik}(\mathcal{U}^R_{ki})^2 + \mu_{ki}\mathcal{U}^R_{ik}\mathcal{U}^R_{ki}\right]$$
$$+ 2(RC_u)_k\left(\int_{\mathbb{R}^d}\xi_k\tau\overline{\nabla}^R_k g\,\mathrm{d}p - 4\Omega_{kkk}\int_{\mathbb{R}^d}\xi_k^2\tau\frac{\partial g}{\partial\xi_k}\,\mathrm{d}p + 2\sum_{i=1}^d\int_{\mathbb{R}^d}\xi_i\xi_k\tau\overline{\nabla}^{\Omega}_{iik}g\,\mathrm{d}p\right)$$
$$+ 2\sum_{i,j=1}^d\left[R_{ij}\frac{\partial}{\partial x_j} - (\Omega_{jji}+\Omega_{jij})\right]\left(\int_{\mathbb{R}^d}\xi_i\xi_k\tau\overline{\nabla}^R_i g\,\mathrm{d}p - 4\Omega_{iii}\int_{\mathbb{R}^d}\xi_i^2\xi_k\tau\frac{\partial g}{\partial\xi_i}\,\mathrm{d}p\right.$$
$$\left.+ 2\sum_{l=1}^d\int_{\mathbb{R}^d}\xi_i\xi_k\xi_l\tau\overline{\nabla}^{\Omega}_{lli}g\,\mathrm{d}p\right) - 4\sum_{i=1}^d(1-\delta_{ik})(\Omega_{kki}+\Omega_{kik})\left(\int_{\mathbb{R}^d}\xi_i\xi_k\tau\overline{\nabla}^R_i g\,\mathrm{d}p\right.$$
$$\left.- 4\Omega_{iii}\int_{\mathbb{R}^d}\xi_i^2\xi_k\tau\frac{\partial g}{\partial\xi_i}\,\mathrm{d}p + 2\sum_{l=1}^d\int_{\mathbb{R}^d}\xi_i\xi_k\xi_l\tau\overline{\nabla}^{\Omega}_{lli}g\,\mathrm{d}p - 4\int_{\mathbb{R}^d}\xi_i\xi_k^2\tau\overline{\nabla}^{\Omega}_{kki}g\,\mathrm{d}p\right)$$
$$- 8\sum_{i,j=1}^d(\Omega_{kij}+\Omega_{kji})\int_{\mathbb{R}^d}(1-\delta_{ij})\xi_i\xi_j\xi_k\tau\overline{\nabla}^{\Omega}_{ijk}g\,\mathrm{d}p$$
$$- 4\sum_{i=1}^d\Omega_{kii}\left(\int_{\mathbb{R}^d}\xi_i\xi_k\tau\overline{\nabla}^R_k g\,\mathrm{d}p - 4\Omega_{kkk}\int_{\mathbb{R}^d}\xi_i\xi_k^2\tau\frac{\partial g}{\partial\xi_k}\,\mathrm{d}p + 2\sum_{l=1}^d\int_{\mathbb{R}^d}\xi_i\xi_k\xi_l\tau\overline{\nabla}^{\Omega}_{llk}g\,\mathrm{d}p\right).$$

As in the proof of Proposition 4.4, using now the first order approximate macroscopic equation

$$\rho\left[R\frac{\partial u}{\partial t} + R(\nabla_x u)u\right]_k + 2\sum_{i=1}^d R_{ki}\frac{\partial W_k}{\partial x_i} - 2\sum_{i=1}^d(\Omega_{iik}+\Omega_{iki})W_k - 2\sum_{i=1}^d\Omega_{kii}W_i = 0$$

and formula (5.28) connecting u and $\bar{u}$ leads to the equation (5.33), (5.34). This concludes the proof. □

5.3. Evolution equation for the rotation matrix

We recall that the kinetic equation under study (in the variable frame) is

$$T_u f = Q(f,u) = \frac{1}{\tau(x,\xi,t)}(\Pi_{\mathcal{N}_{u,R}} f - f),$$

$\Pi_{\mathcal{N}_{u,R}}$ denoting the orthogonal projection (in the velocity variable) onto the space of all functions of the form

$$f(v) = g\left(\frac{(R(v-u))_1^2}{2}, \dots, \frac{(R(v-u))_d^2}{2}\right)$$

and where u and f are connected in such a way that the total momentum operator is preserved (cf. (2.7)). Now the question is how to choose the rotation matrix $R(x,t)$. Two possibilities will be explored below. The first one consists in choosing R such that the stress tensor is preserved, and the second one used a rotational derivative.

5.3.1. Preserving the stress tensor

Here, we consider that $R(x,t)$ is linked with the distribution function f in such a way that the stress tensor is also preserved. One advantage of this approach is the fact that it leads to large systems of conservation equations at the macroscopic level, including equations on the stress tensor coefficients. This is similar to systems of moments derived from the ellipsoidal-BGK model for instance [19]. The derivation of such moment systems in the present case is deferred to a future work.

In the variables associated with the fluid frame $p = R(x,t)(v-u)$, we then have

$$\int p \otimes p \tau^{-1}(\xi)(\Pi F - F)\,\mathrm{d}p = 0, \tag{5.35}$$

with $\xi = (p_1^2/2, \dots, p_d^2/2)$ and $F(p) = f(v)$. This equivalent to

$$\int_{\mathbb{R}^d} p_i p_j \tau^{-1} F\,\mathrm{d}p = 0 \quad \text{if } i \neq j.$$

Now, if we use a Chapman–Enskog expansion, $F = F_0 + \varepsilon F_1$ with $F_0 = \Pi F$, as previously, we get

$$\int_{\mathbb{R}^d} p_i p_j \tau^{-1} F_1\,\mathrm{d}p = \mathrm{O}(\varepsilon) \quad \text{if } i \neq j.$$

We then deduce from the expression of F_1 given by (5.17), after some calculations

$$\left(\mathcal{U}_{i,j}^R + \mathcal{U}_{j,i}^R\right)\int_{\mathbb{R}^d} \xi_i \xi_j \frac{\partial g}{\partial \xi_i}\,\mathrm{d}p = 0, \quad \text{if } i \neq j,$$

where $\mathcal{U}^R$ is given by (5.6). Observing that

$$\int_{\mathbb{R}^d} \xi_i \xi_j \frac{\partial g}{\partial \xi_i}\,\mathrm{d}p = -\frac{1}{2}\int_{\mathbb{R}^d} \xi_j g(\xi)\,\mathrm{d}p < 0, \quad \text{if } i \neq j,$$

we get

$$\mathcal{U}_{i,j}^R + \mathcal{U}_{j,i}^R = 0, \quad \text{if } i \neq j.$$

Now we use the expression of $\mathcal{U}^R$ given by (5.6) and the fact that we have

$$(T_u R)R^{\mathrm{T}} + R(T_u R)^{\mathrm{T}} = T_u\left(RR^{\mathrm{T}}\right) = 0,$$

and obtain

$$\left[R\sigma(u)R^{\mathrm{T}}\right]_{ij} = 0, \quad \text{if } i \neq j,$$

which means that R diagonalizes the symmetric tensor $\sigma(u) = \nabla_x u + \nabla_x u^{\mathrm{T}}$. In other words, the columns of R are eigenvectors of $\sigma(u)$.

5.3.2. Material derivative of $R(x,t)$

Now, we consider the second choice (regarding the matrix R) where the fluid anisotropy must be preserved by the flow. Precisely, we want to write a transport equation on R that preserves the tensorial and the orthogonal characters of R. The so called 'Jaumann Derivative' provides the simplest transport equation satisfying these constraints [20]. It reads

$$\frac{\mathcal{D}R}{\mathcal{D}t} = T_u R + \frac{1}{2}\left(\omega R + R\omega^{\mathrm{T}}\right) = 0, \tag{5.36}$$

with

$$\omega = \nabla_x u - \nabla_x u^{\mathrm{T}}.$$

In other words this equation gives the time rate of change, following a fluid element, in a coordinate frame rotating with the instantaneous fluid angular velocity. This evolution equation needs a value of R at initial time. A possible value may be a matrix R that diagonalizes $\sigma(u)$ at $t=0$.

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